THE DISSECTION OF RECTANGLES INTO SQUARES

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Introduction. We consider the problem of dividing a rectangle into a finite number of non-overlapping squares, no two of which are equal. A dissection of a rectangle R into a finite number n of non-overlapping squares is called a *squaring* of R of *order* n; and the n squares are the *elements* of the dissection. The term "elements" is also used for the lengths of the sides of the elements. If there is more than one element and the elements are all unequal, the squaring is called *perfect*, and R is a *perfect rectangle*. (We use R to denote both a rectangle and a particular squaring of it.) Examples of perfect rectangles have been published in the literature.[1]

Our main results are:

Every squared rectangle has commensurable sides and elements.[2] (This is (2.14) below.)

Conversely, every rectangle with commensurable sides is perfectible in an infinity of essentially different ways. (This is (9.45) below.) (**Added in proof.** Another proof of this theorem has since been published by R. Sprague: Journal für Mathematik, vol. 182(1940), pp. 60–64; Mathematische Zeitschrift, vol. 46(1940), pp. 460–471.)

In particular, we give in §8.3 a perfect dissection of a square into 26 elements.[3]

There are no perfect rectangles of order less than 9, and exactly two of order 9.[4] (This is (5.23) below.)

The first theorem mentioned is due to Dehn, who remarked[5] that the difficulty of the problem is the semi-topological one of characterizing how the elements fit together. This is overcome here in §1 by associating a certain linear graph (the "normal polar net") with each "oriented" squared rectangle. The metrical properties of the squared rectangle are found to be determined by a certain flow of electric current through this network. Accordingly, in §2 we collect the relevant results from the theory of electrical networks. In particular, the elements of the squared rectangle can be calculated from determinants formed from the incidence matrix of the network. In §3, the elements are expressed in a different way, in terms of the subtrees of the network. This leads

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[1] A bibliography is given at the end of this paper. Numbers in square brackets refer to this bibliography.

[2] Cf. [6], p. 319.

[3] This disproves a conjecture of Lusin; cf. [10], p. 272. For an independent example of a perfect square (published while this paper was in preparation) see [13].

[4] Partly confirming and partly disproving a conjecture of Toepken (see [18]).

[5] [12], p. 402.

to some relations between determinants and the subtrees of a network, and to some duality theorems. In §4, these duality theorems are applied to prove the converse of §1: that to any "polar net" corresponds a squared rectangle; and moreover, it is shown that (roughly speaking) the networks which correspond to the same squared rectangle in its two orientations are dual. In §5, the polar net is used to determine all the squared rectangles of a given order; in particular, the "simple" perfect rectangles of orders <12 are tabulated. §6 contains some theorems on the factorization properties of the elements of a squared rectangle, as determined in §2; as corollaries, we have some sufficient conditions for a squared rectangle to be perfect ((6.20), (6.21)). In §7, we give "non-uniqueness" constructions—in §7.1, of rectangles which can be dissected into the same elements in essentially different ways, and, in §7.2, of pairs of squared rectangles having the same shape but different elements. These constructions depend mostly on considerations of symmetry or duality in the corresponding networks. In §8, the results of §7.2 are used to give "perfect" squares; and in §9, a whole family of "totally different" perfect squares is worked out, and this leads to the result that every rectangle whose sides are commensurable is perfectible.

We conclude (§10) by outlining some generalizations—notably "rectangled rectangles", squared cylinders and tori, "triangulated" equilateral triangles, and "cubed cubes". We prove in particular that no "perfect" dissection of a rectangular parallelopiped into cubes is possible.[6]

1. The net associated with a squared rectangle

1.1. In any squaring of a rectangle R,[7] the sides of all the elements and of R will clearly be parallel to two perpendicular lines. We *orient* R by choosing one of these lines to be "horizontal" (i.e., parallel to the x-axis). The distinction between this configuration, and its reflections in the coördinate axes, is unimportant; but it is convenient to distinguish it from R in the other orientation (obtained by rotating R through an angle of $\frac{1}{2}\pi$), called the *conjugate* of R.

Consider the point-set formed by the horizontal sides of the elements of R. Its connected components will be horizontal line-segments (each consisting of a set of horizontal sides of elements of R); enumerate them as $p_1, \cdots, p_N$, say, where p_1, p_N are the upper and lower edges of R. Take N points $P_1, \cdots, P_N$ in the plane. Let E be an element of R; its upper edge will lie in some one of $p_1, \cdots, p_N$, say p_i: similarly, its lower edge will lie in p_j $(i \neq j)$. Join the points P_i, P_j by a line (simple arc) e. By taking all elements E of R, we get a network (linear graph) on $P_1, \cdots, P_N$ as vertices and the e's as 1-cells. Figure 1 provides an example.

The points P_1, P_N are the *poles* of the network. We can arrange the joins e in such a way that

[6] Answering a question raised by Chowla in [5].

[7] Throughout, all squares are supposed to have positive sides; thus zero elements are excluded.

(1.11) the network is realizable in a plane with no two 1-cells intersecting (except at a vertex).

(1.12) No circuit encloses a pole.

For we can realize the network as follows. Take P_i to be the mid-point of p_i ; and take $\epsilon > 0$ sufficiently small. For each element E, take the vertical segment which bisects E, and cut off a length ϵ from each end, leaving a segment AB, say. Join the upper end of AB, A, to the P_i corresponding to the upper boundary of E, by a straight line-segment, and similarly join B to P_j corresponding to the lower edge of E. The path P_iABP_j is defined to be e. It is now easily verified that (1.11) and (1.12) hold.

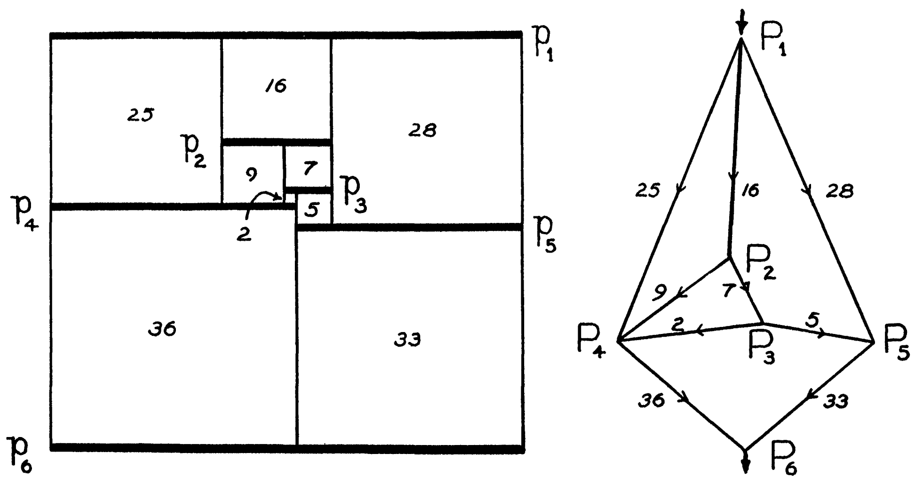

FIG. 1

Also we have clearly

(1.13) The network is connected.

Remark. In general there may be several 1-cells joining two vertices, though not if the squaring is perfect.

(1.14) DEFINITIONS. A network with more than one vertex, satisfying (1.11) and (1.13), is called a *net.* If two of the vertices of a net are assigned as "poles", and (1.12) is satisfied, the net is a *polar net* (p-net). The network constructed above is the *normal polar net* of the squared rectangle.

1.2. **Kirchhoff's laws.** With each 1-cell $e = P_iP_j$ of our normal p-net, associate the length of the side of the corresponding element E, directed from the "upper" point (P_i) to the "lower" point (P_j); call this the *current* in e. Then

(1.21) Except at the poles, the total current flowing into P_i is zero.

(For current flowing in = length of p_i = current flowing out.)

(1.22) The algebraic sum of the currents round any circuit is zero.

(For the current in a "wire" $e = P_iP_j$ is the vertical height of p_i above p_j.)

(1.23) The sum of the currents flowing into P_1 = length of horizontal side of R = sum of the currents flowing out of P_N.

(1.21) and (1.22) are the usual Kirchhoff laws for a flow of electric current in the net from P_1 to P_N, it being assumed that each 1-cell is a wire of unit conductance.

["Rectangulations" of rectangles can be dealt with similarly; the conductance of e will then be the ratio of the sides of E.]

Equations (1.21) and (1.22) can be interpreted differently. Consider the cellular 2-complex formed by embedding our p-net in a 2-sphere. We have on it a *Kirchhoff chain* (K-chain), viz., the 1-chain Σ (current in e)$\cdot e$. Then

(1.24) The K-chain is a cycle modulo its poles. (This $\leftrightarrow$ (1.21).)

(1.25) The K-chain is an absolute cocycle. (This $\leftrightarrow$ (1.22).)

2. Some results from the electrical theory of networks

2.1. In the previous section, we reduced the study of squared rectangles to the study of certain flows of electricity in networks. Here we collect the results on electrical networks in general which will be useful later.

Let $\mathfrak{N}$ be a connected network whose vertices are $P_1, \cdots, P_N$ ($N \geqq 2$). The 1-cells are called *wires*; there may be more than one wire joining two vertices, and there may be wires whose two ends coincide. With each wire is associated a positive real number, its *conductance*. We define a matrix $\{c_{rs}\}$ as follows:

(2.11) If $r \neq s$,

$$-c_{rs} = \begin{cases} \text{sum of conductances of all wires joining } P_r, P_s, \\ 0 \text{ if there are no such wires;} \end{cases}$$

$$c_{rr} = \text{sum of conductances of all wires joining } P_r \text{ to } \textit{other} \text{ vertices.}$$

Thus

$$c_{rs} = c_{sr}, \qquad \sum_r c_{rs} = 0. \tag{2.12}$$

We make the convention that if $\mathfrak{N}$ is explicitly called a *net*, all its conductances are 1. (The matrix $\{-c_{rs}\}$ is then the product of the usual incidence matrix of the oriented network, with its transpose.)

Let us return to the general case; from (2.12) we can readily show that all first cofactors of $\{c_{rs}\}$ are equal. We call their common value the *complexity* of the network, and denote it by C. It is known that $C > 0$. (An independent proof is given below; see (3.14).)

The second cofactor obtained by taking the cofactor of the component c_{su} in the cofactor of c_{rt} ($r \neq s$, $t \neq u$) is denoted by $[rs, tu]$. (If $N = 2$, $[12, 12] = 1 = -[21, 12]$.) We put $[rr, tu] = 0 = [rs, tt]$. The $[rs, tu]$'s are called the *transpedances* (generalized transfer impedances) of $\mathfrak{N}$.

Consider a flow of current from P_x to P_y (the *poles*). The currents in the wires satisfy (1.21); the potential differences (P.D.'s) satisfy the analogue of (1.22); and the total current I is given by (1.23). It is known[8] that these conditions (with Ohm's Law) determine the flow *uniquely* when I is given, and that

(2.13) P.D. from P_r to P_s when current I enters at P_x and leaves at P_y is $[xy, rs]\cdot I/C$.

It is convenient to take $I = C$, thus fixing the values of the currents and P.D.'s of the network. The flow with $I = C$ is called the *full flow*; and we speak of the "full currents", etc.

Applying this to the normal p-net of a squared rectangle, where all conductances are 1, so that all the transpedances are integers, we see from (1.21)-(1.23) and (2.13) that

(2.14) *Every squared rectangle has commensurable sides and elements.*

The H.C.F. of the full currents of a p-net is the *reduction* ρ of the p-net. Notice that ρ is also the H.C.F. of all the full P.D.'s of the p-net. The flow with $I = C/\rho$ is the *reduced* flow.

2.2. **Properties of the transpedances.** We have

(2.21) $$[rs, tu] = [tu, rs] = -[sr, tu],$$

(2.22) $$\sum_x c_{tx}\cdot[rs, tx] = C\cdot(\delta_{ts} - \delta_{tr}),$$

(2.23) $$[rs, tu] + [rs, uv] = [rs, tv].$$

(2.22) and (2.23) verify that (2.13) does in fact provide a solution of the Kirchhoff equations, and that the current at each pole is C.

We call $[rs, rs]$ the *impedance* of r, s, and write it $V(rs)$. Then

(2.24) $$V(rs) = V(sr), \qquad V(rr) = 0,$$

(2.25) $$2\cdot[rs, tu] = V(ru) + V(st) - V(su) - V(rt) \qquad \text{(from (2.23))},$$

(2.26) $$[rs, tu] + [tr, su] + [st, ru] = 0.$$

2.3. **Alterations to the network.** For later use, we need to know the effect on the transpedances of making certain alterations to the network $\mathfrak{N}$.

I. Introduce a new wire joining a vertex P_m of $\mathfrak{N}$ to a new vertex P_0. Let the new wire have conductance c; then, in the new network $\mathfrak{N}_1$,

$$C_1 = cC, \qquad V_1(m0) = C;$$

(2.31) $$[ab, xy]_1 = c\cdot[ab, xy] \quad \text{if } 0 \neq a, b, x, y; \qquad [ab, m0]_1 = 0;$$

$$V_1(x0) = V_1(xm) + V_1(m0) = c\cdot V(xm) + C.$$

[8] [8], pp. 324–331.

These results are immediate from the definitions.

II. Identify two points P_x, P_y and ignore any wire that may have joined them. In the new network $\mathfrak{N}_2$,

$$C_2 = [xy, xy] = V(xy) \qquad \text{(from the definitions)}, \tag{2.32}$$

$$[rs, tu]_2 = \frac{[rs, tu]\cdot V(xy) - [rs, xy]\cdot[tu, xy]}{C} \tag{2.33}$$

(for these expressions satisfy Kirchhoff's laws for $\mathfrak{N}_2$, and agree with (2.32)). In particular,

$$V_2(rs) = \frac{V(rs)\cdot V(xy) - [rs, xy]^2}{C}. \tag{2.34}$$

((2.33) may be generalized as follows: C^n *divides the* $(n + 1)$*-th order determinants formed as minors of the matrix of transpedances.* This is an extension of the Cauchy-Sylvester identity.[9])

III. Introduce a new wire of conductance c in $\mathfrak{N}$, joining P_x and P_y. In the new network $\mathfrak{N}_3$ we have, from their definitions as determinants,

$$C_3 = C + c\cdot V(xy) = C + c\cdot C_2 \qquad \text{(from (2.32))}; \tag{2.35}$$

$$[rs, xy]_3 = [rs, xy]; \qquad \text{in particular, } V_3(xy) = V(xy). \tag{2.36}$$

Also

$$[rs, tu]_3 = [rs, tu] + c\cdot[rs, tu]_2; \tag{2.37}$$

for III is a combination of I and II. We introduce a new vertex P_0, join it to P_x by a wire of conductance c, and identify P_y and P_0. This enables us to verify (2.37).

3. Subtrees of a network: duality

We shall now characterize the complexity (and hence the transpedances) of a network more topologically, in terms of the "subtrees" of the network. This enables us to prove some duality theorems which will be useful later (§4) and are of interest in themselves.

3.1. As in the previous section, let $\mathfrak{N}$ be a connected network with conductances. By a *subnetwork* $\mathfrak{M}$ of $\mathfrak{N}$, we mean a network consisting of *all* the vertices of $\mathfrak{N}$ and some (or all) of the wires of $\mathfrak{N}$. A *subtree* of $\mathfrak{N}$ is a subnetwork which is a "tree"; i.e., is connected and has no circuits. Enumerate all the subtrees of $\mathfrak{N}$; let M_r be the product of the conductances of the wires of the r-th tree. Define H by:

$$H = \sum_r M_r. \tag{3.11}$$

[9] [19], p. 87.

When a new wire of conductance c is inserted joining P_x, P_y, let "H" for the new network be H_3; and when P_x, P_y are identified (as in §2.3, II), let "H" become H_2. Clearly,

$$H_3 = H + c \cdot H_2 . \tag{3.12}$$

But this is the relation which holds between the complexities of these networks (2.35).

Also, for a connected network with only two vertices, C = sum of conductances of the wires joining P_1 to $P_2 = H$. Hence, by induction on the numbers of vertices and wires in $\mathfrak{N}$, we have:[10]

(3.13) **Theorem.** *For any connected network with more than one vertex, having conductances assigned to the* 1*-cells,* $C = H$.

If the conductances are all positive, we clearly have $H > 0$. This proves

$$C > 0. \tag{3.14}$$

This interpretation of complexity in terms of trees enables us, if (2.32) is used, to express $V(xy)$ in terms of the trees of networks formed from $\mathfrak{N}$ by identifying certain pairs of its vertices, and hence in terms of the "tree-pairs" of $\mathfrak{N}$ (formed by omitting one wire from a subtree). Hence, using (2.25), we can get similar interpretations for all the transpedances.

In the case of a net, all conductances are 1, so H = number of subtrees of $\mathfrak{N}$; thus (3.13) gives an explicit formula for the number of subtrees of any connected network, in terms of the incidences of the network.

3.2. **Duality relations.** Now suppose that $\mathfrak{N}$ can be imbedded in a 2-sphere, and let $\mathfrak{N}^*$ be its dual on the sphere. The conductivity of a wire of $\mathfrak{N}^*$ is defined to be the reciprocal of that of the dual wire of $\mathfrak{N}$. Thus $\mathfrak{N}^{**} = \mathfrak{N}$, and the dual of a net is a net. The *codual* of a subnetwork $\mathfrak{M}$ of $\mathfrak{N}$ is the subnetwork $\mathfrak{M}^c$ of $\mathfrak{N}^*$ whose 1-cells are those *not* dual to any wire of $\mathfrak{M}$. Clearly $\mathfrak{M}^{cc} = \mathfrak{M}$.

It can be shown that

(3.21) A subnetwork $\mathfrak{M}$ of $\mathfrak{N}$ is a tree if and only if both $\mathfrak{M}$ and $\mathfrak{M}^c$ are connected.

Hence

(3.22) If $\mathfrak{M}$ is a subtree of $\mathfrak{N}$, then $\mathfrak{M}^c$ is a subtree of $\mathfrak{N}^*$; and conversely.

Let M_r^* equal the product of conductances of wires in the subtree (of $\mathfrak{N}^*$) which is codual to the r-th subtree of $\mathfrak{N}$. Let ω equal the product of conductances of all wires of $\mathfrak{N}$. Then, clearly,

$$M_r = \omega \cdot M_r^* . \tag{3.23}$$

[10] This result is due in principle to Kirchhoff ([9], p. 497). Cf. also [3].

Hence, using (3.22), (3.11), and (3.13), we have

(3.24) If C^* is the complexity of the dual of $\mathfrak{N}$, $\omega \cdot C^* = C$.

In particular, we have proved

(3.25) THEOREM. *Dual nets have equal complexities.*

3.3. **Polar duality.** Let $\mathcal{P}$ be a p-net. By (1.12), we can join the poles of $\mathcal{P}$ by an extra wire e_0, without violating (1.11). The resulting net $\mathcal{C}$ is called the *completed* net (c-net) of $\mathcal{P}$. Let $\mathcal{C}$ be imbedded in a 2-sphere, and let $\mathcal{C}^*$ be the dual of $\mathcal{C}$. From $\mathcal{C}^*$ omit e_0^*, the dual of e_0, and take the ends of e_0^* as poles. We get a p-net $\mathcal{P}'$, the *polar dual* of $\mathcal{P}$.[11]

Clearly $\mathcal{P}'' = \mathcal{P}$.

(The importance of polar duality arises from the fact that, as we shall show in §4.3, polar dual p-nets correspond to the same squared rectangle in its two "orientations" (§1.1).)

The p-dual (polar dual) of any 1-chain on $\mathcal{P}$ is defined in the obvious way (as having the same multiplicity on e_i^* as the given chain has on e_i).

(3.31) THEOREM. *The p-dual of the full Kirchhoff chain on a p-net $\mathcal{P}$ is the full Kirchhoff chain on the p-dual p-net $\mathcal{P}'$.*

Proof. We use $\mathcal{S}\mathfrak{N}$ to denote the cellular 2-complex formed by a network $\mathfrak{N}$ imbedded in a 2-sphere. F, δ are (as usual) boundary and coboundary operators, and * denotes duality with respect to the 2-sphere.

By (1.24), (1.25), the full K-chain $\mathcal{K}$ on $\mathcal{P}$ is a cycle relative to P_1, P_N (the poles of $\mathcal{P}$), and an absolute cocycle on $\mathcal{S}\mathcal{P}$. Hence, in $\mathcal{S}\mathcal{C}$ (where $\mathcal{C}$ is the completed net of $\mathcal{P}$) $\mathcal{K}$ is

(i) a relative cycle mod P_1, P_N, and

(ii) a relative cocycle mod the two 2-cells, say σ_1, σ_2, which have incidence with e_0, the "extra" join.

Dualizing, in $\mathcal{S}\mathcal{C}^*$, we see that $\mathcal{K}^*$ is

(i) a relative cocycle mod the 2-cells P_1^*, P_N^* and

(ii) a relative cycle mod σ_1^* and σ_2^*, the poles of $\mathcal{P}'$.

But $\mathcal{K}^*$ has zero multiplicity on e_0^*, for $\mathcal{K}$ has zero multiplicity on e_0. Hence $\mathcal{K}^*$ is (from (i)) a cycle on $\mathcal{P}'$ mod its poles, and (from (ii)) a cocycle on $\mathcal{S}\mathcal{P}'$ mod the 2-cell consisting of P_1^* and P_N^* together. But a single 2-cell cannot be a coboundary; for, dualizing, this would require a single vertex to be a boundary. Hence $\mathcal{K}^*$ is an absolute cocycle on $\mathcal{S}\mathcal{P}'$, besides being a cycle mod its poles. So $\mathcal{K}^*$ is a K-chain on $\mathcal{P}'$.

Let $\mathcal{K}'$ be the full K-chain on $\mathcal{P}'$; thus $\mathcal{K}^* = k \cdot \mathcal{K}'$, for some k.

[11] There may be several ways of placing e_0 on the sphere, and consequently several polar duals of $\mathcal{P}$ (differing, however, only trivially). We suppose that one of these is chosen arbitrarily. In the open plane, a convention will be introduced to make $\mathcal{P}'$ unique; cf. §§4.2, 4.3.

Let $\mathcal{P}$ have complexity C, and $V(1N) = V$. Let the corresponding numbers for $\mathcal{P}'$ be C', V'. Using (2.22) (with $c_{tx} = 1$), we have, in $\mathcal{P}$,

$$F(\mathcal{K}) = C \cdot (P_1 - P_N).$$

Therefore, in $\mathcal{SP}'$, $\delta(\mathcal{K}^*) = C \cdot (P_1^* - P_N^*)$ (these cells being oriented suitably). So

$$C = \begin{cases} \text{sum of currents around } F(P_1^*) \text{ in } \mathcal{K}^*, \\ \text{sum of currents along a path joining the end-points of } e_0^*, \\ \text{total P.D. between the poles of } \mathcal{P}', \text{ in the flow } \mathcal{K}^*. \end{cases}$$

Thus

(3.32) $$C = k \cdot V'.$$

Similarly,

(3.33) $$C' = (1/k) \cdot V.$$

Now, by (2.35), the complexity of $\mathcal{C}$ is $C + V$. Similarly, the complexity of $\mathcal{C}^*$ is $C' + V' = k \cdot (C + V)$, by (3.32), (3.33). But by (3.25) these complexities are equal. Hence $k = 1$, and $\mathcal{K}^*$ is the *full* K-chain on $\mathcal{P}'$.

4. The correspondence between p-nets and squared rectangles

4.1. We now sketch a proof showing that to each p-net corresponds a squared rectangle. This correspondence is many-one and is clarified by introducing the "normal form" of a p-net (§4.2). We can then set up a 1-1 correspondence between classes of p-nets (having the same normal form) and "oriented" squared rectangles, and can prove that p-dual p-nets correspond to "conjugate" squared rectangles. (Cf. §1.1.)

(4.10) LEMMA. *For a K-chain in a p-net $\mathcal{P}$, whose poles are P_1, P_N (suitably numbered),*

(4.11) *the potential of each vertex lies between the potentials of the poles;*

(4.12) *no currents go into P_1, or out of P_N;*

(4.13) *at a vertex P_i, there is an angle (in the plane) containing all ingoing currents, whose reflex contains all outgoing currents;*

(4.14) *on the boundary of a 2-cell of $\mathcal{SP}$, there are two vertices P_i, P_j such that no current round this boundary goes from P_j towards P_i.*

(We make the convention that zero currents do not go in or out.)

Proof. Let P_i be any vertex, and suppose a current goes into P_i. Then a current goes out of P_i along at least one wire, ending at P_j, say; and so on, until we reach a pole P_N (say). All this time the potential has been falling, so P_N is eventually reached; and the potential of P_i is thus not less than that of P_N. If all the currents at P_i are zero, we can connect P_i to a vertex P_k at which not all currents are zero, by a path of zero currents; and P_i, P_k have the same potential. Thus in all cases the potential of P_i is not less than that of P_N; and similarly it is not greater than that of P_1. This proves (4.11).

(4.12) follows at once from (4.11).

(4.13) has been proved for the poles; so let $i \neq 1$, N, and suppose that two outgoing currents at P_i separate (in the plane) two ingoing ones. As in the proof of (4.11), we can continue each of the first two wires into a path down to P_N, along which the current falls; and similarly we can extend the other two wires into paths of rising potential up to P_1. Hence one of the two former paths must intersect one of the latter again, say in P_j ($i \neq j$). The potential of P_j is both less than and greater than the potential of P_i. This is a contradiction, and so (4.13) is proved.

(4.14) follows from (4.13) and (4.12) by dualizing, if we use (3.31).

4.2. **Normal form of a p-net.** Let $\mathcal{P}$ be a p-net imbedded in the open plane in such a way that its poles, P_1, P_N, can be joined in the "outside region" of $\mathcal{SP}$. (That is, $\mathcal{P}$ is first imbedded in the closed 2-sphere, an extra join e_0 of the poles is inserted, and the "point at infinity" is then taken to be in the 2-cell of $\mathcal{SP}$ which contains e_0.) We define the *normal form* of $\mathcal{P}$, as so placed in the plane, as follows:

Consider any (not identically zero) K-chain $\mathcal{K}$ on $\mathcal{P}$. Some currents may be zero; delete the corresponding wires, and delete all vertices at which all currents are zero. Since $C > 0$, we are left with a p-net still, having P_1, P_N as poles. Using (2.31), (2.37), (2.36) (with $c = 1$), we see that $\mathcal{K}$ is a K-chain for the new p-net $\mathfrak{N}$. Next take each *finite* 2-cell of $\mathcal{S}\mathfrak{N}$, and consider the vertices on its boundary. By (4.14), the 2-cell with its boundary is homeomorphic to a convex polygon which has one highest point and one lowest point, and in which the potentials of the vertices increase with their heights. Moreover, they increase *strictly*; for now no currents are zero. Hence equipotential vertices on this boundary occur at most in pairs, which can all be respectively identified by a deformation across the 2-cell. Making all these identifications for all the finite 2-cells, we end with a p-net $\mathfrak{N}_0$, on P_1, P_N as poles, on which $\mathcal{K}$ is still a K-chain (by (2.33)). And there are now no two vertices at the same potential which can be joined without crossing some wire of $\mathfrak{N}_0$, or separating the poles in the "outside" region. In particular, there are no zero currents. $\mathfrak{N}_0$ is called the *normal form* of $\mathcal{P}$, in its given imbedding in the plane.

Notice that, while we have proved that $\mathcal{P}$, $\mathfrak{N}_0$ have the same *reduced* K-chains, they need not have the same full K-chains.

It is easily seen that the normal p-net of a squared rectangle is its own normal form.

4.3. We next prove

(4.31) Theorem. *To every p-net $\mathcal{P}$ in the open plane corresponds a squared rectangle R, whose normal p-net is the normal form of $\mathcal{P}$. Polar dual p-nets correspond to conjugate squared rectangles.*

(The polar dual of a p-net $\mathcal{P}$ in the open plane is itself put in the open plane in the obvious way—e_0^* is taken to be in the "outside".)

Proof. Consider the full K-chain $\mathcal{K}$ on $\mathcal{P}$ and its dual, the full K-chain on the p-dual net $\mathcal{P}'$. (By (3.31).) Let $\mathcal{P}$ have complexity C, and let the P.D. between its poles be V $(= V(xy))$. Thus ((3.32), (3.33)) the analogous numbers for $\mathcal{P}'$ are V and C respectively. We can take the lowest potentials in $\mathcal{P}$ and $\mathcal{P}'$ to be zero. Suppose a wire e in $\mathcal{P}$ has its end-points at potentials V_1, V_2, and its dual e^* has its end-points at potentials V_1', V_2'. If μ is a number such that $V_1 < \mu < V_2$, we say that *e comprises* $(\ , \mu)$; and if λ is such that $V_1' < \lambda < V_2'$, then *e comprises* $(\lambda, \ )$. If both relations are true, we say that *e comprises* (λ, μ).

Now, observing that $V_2 - V_1 =$ current in $e =$ current in $e^* = V_2' - V_1'$, we construct a squared rectangle R as follows: In a rectangle of height V and base C, we take, for each wire e of $\mathcal{P}$, the (closed) *square E* whose horizontal sides are at a height V_1, V_2 above the base (x-axis) and whose vertical sides are at a distance V_1', V_2' to the right of the left-hand vertical side (y-axis). If the current in e is zero, this square reduces to a single point, and is omitted.

Let $\lambda \neq$ any potential of a vertex of $\mathcal{P}'$, and $\mu \neq$ any potential of a vertex of $\mathcal{P}$. Then, if $0 < \lambda < C$, and $0 < \mu < V$, we have the following:

The wires (of $\mathcal{P}$) comprising $(\lambda, \ )$ form a single path from pole to pole, along which the direction of the current is constant. For, by (4.12) and duality, there is just one such wire terminating at each pole; and from (4.14), if one such wire carries current to a vertex, then just one such wire carries current from that vertex, and no more such wires terminate at that vertex.

Along this path, the potential increases steadily from pole to pole; also, by choice of λ, the currents along the path are non-zero. Hence just one wire in it comprises $(\ , \mu)$. So just one wire of $\mathcal{P}$ comprises (λ, μ). Thus the point of coördinates (λ, μ) belongs to just one of the squares E. It follows that the whole rectangle is filled completely and without overlap (except of boundaries of squares).

It is easy to see that the normal p-net of the squared rectangle so constructed is—to within reflection in the axes (which we always disregard)—the normal form of $\mathcal{P}$. Also, it is clear from the construction that the squared rectangle assigned to $\mathcal{P}$ differs from that assigned to $\mathcal{P}'$ only by interchange of horizontal and vertical; i.e., the two squared rectangles are conjugate.

In this way, we have a 1-1 correspondence between classes of p-nets in the plane having the same normal form, and "oriented" squared rectangles.

DEFINITIONS. As suggested by (4.31), the complexity of a p-net is called its (full) *horizontal side* (often written H instead of C); and the full P.D. between its poles is its *vertical side* (V). The "full elements" and "full sides" of a squared rectangle refer to those of its normal p-net. The "reduced elements" will be the same for all corresponding p-nets.

4.4. Defining a *cross* as a point of a squared rectangle which is common to four elements, and an "uncrossed" squared rectangle as one which has no crosses, we have:

The normal p-nets of uncrossed conjugate squared rectangles are p-duals.

For let $\mathcal{P}$ be the normal p-net of the squared rectangle R; and let $\mathcal{P}'$ be the p-dual of $\mathcal{P}$. Let $\mathcal{Q}$ be the normal p-net of the conjugate R' of R; thus, from

§4.3, $\mathcal{Q}$ is the normal form of $\mathcal{P}'$. Now, in deriving the normal form of $\mathcal{P}$ (as in §4.2) there are no zero currents to suppress; and there are no identifications of vertices possible, as otherwise R', and hence R, would have a cross. So $\mathcal{P}' = \mathcal{Q}$. That is, $\mathcal{P}$ and $\mathcal{Q}$ are p-duals.

(This result could be extended to crossed squared rectangles by making a suitable convention modifying the normal p-net when crosses are present; e.g., by regarding a cross as an "element of side zero".)

5. Enumeration of squared rectangles

5.1. **Computation.** To find all the squared rectangles of a given order n, we have only to make a list of all p-nets having n wires. There is no difficulty in this, if n is not too large. We can save some labor by noting that p-dual nets give essentially the same rectangles; also we can assume that no part of a net, not containing a pole, is joined to the rest only at one vertex. (For the currents in this part would all be zero, whereas we can restrict ourselves to "normal forms".) A convenient way of carrying out the calculations is to consider the c-nets. From each net of $n + 1$ wires, we remove one wire and take its end-points as poles in the remaining net (if it is a net; i.e., is connected). Dual c-nets give rise to pairs of polar dual p-nets; so we need consider only half the c-nets. The working can be simplified by a proper use of §2.2. In practice, the Kirchhoff equations are best solved directly (without using determinants); a single determinant then gives the *full* elements for all the p-nets derived from one c-net.

It follows from §2.3 that all p-nets derived as above from the same c-net will have the same (full) semiperimeter, viz., the horizontal side of the c-net; and that two p-nets which differ only in the choice of poles, and their (non-polar) duals, all have the same (full) horizontal sides, viz., the complexity of the nets. (By (3.25).) Thus a number which appears in the $(n + 1)$-th order as a side appears (several times) in the n-th order as a semiperimeter. These facts are illustrated in the table below (§5.3).

5.2. **The perfect rectangles of least order. "Simple" perfect rectangles**

(5.21) A squared rectangle which contains a smaller squared rectangle (and any p-net corresponding to it) is called *compound*; all other squared rectangles and p-nets are *simple*. A p-net $\mathcal{P}$, without zero currents, which has a part $\mathcal{Q}$ such that $\mathcal{Q}$ contains more than one wire, $\neq \mathcal{P}$, is joined to the rest at only two vertices Q_1, Q_2, and contains no pole (except perhaps for Q_1 or Q_2) is compound. For $\mathcal{Q}$ must be connected; and the squared rectangle corresponding to $\mathcal{P}$ will contain the smaller squared rectangle which corresponds to $\mathcal{Q}$ (with Q_1, Q_2 as poles).

(5.22) *"Trivial" imperfection.* If a p-net has two equal non-zero currents, it is *imperfect*, and these currents constitute an "imperfection". (This is equivalent to saying that the corresponding squared rectangle is not perfect.) If a p-net has a part, not containing a pole, joined to the rest by only two wires, or if it has a pair of vertices joined by two (or more) wires, these two wires will clearly have equal currents. If these currents are non-zero, the resulting imperfection is said to be *trivial.* A p-net which has a non-trivial imperfection is called *non-trivially imperfect.* A non-trivially imperfect p-net may or may not have a trivial imperfection.

We now have the theorem:

(5.23) *The c-net derived from a simple perfect rectangle has no part (consisting of more than one wire and of less than all but one wire) joined to the rest at less than three vertices; and the same is true of its dual.*

For the normal p-net of the simple perfect squared rectangle (or of the conjugate squared rectangle) will otherwise have a zero current, or a trivial imperfection, or be compound.

A perfect rectangle of the smallest possible order must evidently be simple. Applying (5.23) to the method of §5.1, we readily find that

There are no perfect rectangles of order less than 9, *and exactly two perfect rectangles of order* 9.

Of the latter, one is well known;[12] the other is, we believe, new and has been drawn in Figure 1.

Below, we give a list of the simple perfect rectangles of orders 9–11. The compound perfect rectangles of these orders follows trivially.

5.3. **Table of simple perfect rectangles.**

Order	Full Sides	Semi-perimeter	Description of Polar Net (current from P_a to P_b = ab)	Reduction
9	66, 64	130	$ab = 30, ac = 36, bd = 14, cd = 8, be = 16, de = 2, ef = 18, df = 20, cf = 28.$	2
	69, 61	130	$ac = 25, ab = 16, ae = 28, bc = 9, bd = 7, dc = 2, de = 5, cf = 36, ef = 33.$	1
10	114, 110	224	$ab = 60, ac = 54, cb = 6, ce = 22, cd = 26, be = 16, ed = 4, bf = 50, ef = 34, df = 30.$	2
	130, 94	224	$ab = 44, ac = 38, ae = 48, cb = 6, ce = 10, cd = 22, ed = 12, bf = 50, df = 34, ef = 46.$	2
	104, 105	209	$ab = 60, ac = 44, cb = 16, cd = 28, bd = 12, be = 19, de = 7, bf = 45, ef = 26, df = 33.$	1
	111, 98	209	$ab = 44, ad = 26, ae = 41, dc = 11, de = 15, ce = 4, cb = 7, eb = 3, bf = 54, ef = 57.$	1
	115, 94	209	$ab = 34, ac = 19, ad = 23, ae = 39, cb = 15, cd = 4, de = 16, db = 11, bf = 60, ef = 55.$	1
	130, 79	209	$ab = 34, ac = 23, ad = 35, ae = 38, cb = 11, cd = 12, de = 3, bf = 45, df = 44, ef = 41.$	1

[12] First found, apparently, by Moroń [11]. See also [10], p. 272; [2], p. 93; [14], p. 8; and [4].

The full sides and semiperimeters of the simple perfect rectangles of the 11-th order are:

Order	Semi-perimeter	Sides
11	336	127, 209; 151, 185
	353	144, 209; 159, 194; 162, 191; 166, 187; 168, 185; 176, 177
	368	159, 209; 169, 199; 172, 196; 177, 191; 183, 185
	377	168, 209; 178, 199; 183, 194
	386	162, 224; 177, 209; 181, 205; 190, 196; 191, 195; 192, 194

Four of these are reducible, with reduction = 2; these are the rectangles whose sides are both even.

Of the 67 simple perfect rectangles of the 12-th order, eleven have reduction 2, eight have reduction 3, and one has reduction 4.

6. Theorems on reduction

In perfect rectangles of higher orders, much larger reductions occur; for example, a 19-th order rectangle with reduced sides 144 and 155 has $\rho = 80$. Its reduced elements are: $ab = 46$, $ad = 40$, $af = 28$, $ag = 41$, $bc = 10$, $bi = 36$, $ci = 26$, $dc = 16$, $de = 3$, $dh = 21$, $eh = 18$, $fe = 15$, $fg = 13$, $gk = 54$, $hl = 39$, $ij = 62$, $kj = 49$, $kl = 5$, $lj = 44$.

6.1. The following theorems on reduction are of interest.

(6.11) Theorem. *If one of the currents in a p-net is zero, the net is reducible.*

Let the poles be P_r, P_s, and the zero current be in a wire joining P_x, P_y. Then the transpedance $[rs, xy]$ is zero. On removing the wire in question (use (2.37) with $c = -1$, and (2.33)), the new value for $[rs, tu]$ is

$$[rs, tu]' = [rs, tu] - \frac{[rs, tu]\cdot V(xy) - [rs, xy]\cdot[tu, xy]}{C}$$

$$= [rs, tu]\cdot\frac{C - V(xy)}{C}.$$

Now, $C > C - V(xy) = C'$ (by (2.35)) > 0 (by (3.14)). Hence the H.C.F. of the $[rs, tu]$'s must be at least $C/(C - V(xy)) > 1$.

Definition. Let a positive integer $n = m\cdot k^2$, where m is square-free. Then k is called the *lower square root* of n, and mk is the *upper square root.*

(6.12) Theorem. *Let the full sides of a p-net be H, V. Then the reduction ρ is a multiple of the upper square root of the H.C.F. of H and V.*

By (2.34), remembering that $V_2(rs)$ is an integer, we have

$$C \text{ divides } V(rs)\cdot V(xy) - [rs, xy]^2.$$

Since $C = H$, and $V(rs) = V$ (taking P_r, P_s as poles), it follows that the H.C.F. of H, V divides $[rs, xy]^2$; whence the result.

(6.13) COROLLARY. *If the reduced sides of a squared rectangle have H.C.F. σ, then the reduction of any corresponding p-net is divisible by σ.*

(For, by (6.12), σ is a factor of the lower square root of the H.C.F. of the full sides and hence—since the lower square root divides the upper—of ρ.)

(An example is the rectangle 96×99 given in §7.1.)

(6.14) COROLLARY. *Any p-net of a squared square has for reduction a multiple of its reduced side.*

(6.15) COROLLARY. *A necessary and sufficient condition that a p-net be irreducible is that its two full sides be coprime.*

(6.16) THEOREM. *All non-trivially imperfect p-nets are reducible.*

(6.17) LEMMA. *If H, V, k are positive integers such that, for each positive integer n, $(H + nV, k) > 1$, then H, V, k all have a common factor greater than* 1.

Proof. Let N_0 be the product of all the primes which divide k but not H. (Empty product = 1.) Let p_0 be a prime factor of $(H + N_0V, k)$. Suppose $p_0 \nmid H$. Then $p_0 \mid N_0$. Hence, since $p_0 \mid (H + N_0V)$, we have $p_0 \mid H$, and this is a contradiction. So $p_0 \mid H$. Therefore p_0 divides N_0V but not N_0; so that p_0 divides V as well as H and k.

Proof of (6.16). Now let $\mathcal{P}$ be a p-net with full sides H $(= C)$ and V $(= V(1N))$; and let a non-trivial imperfection be $[1N, ab] = [1N, pq] = k$, say. Thus $k > 0$, and we do not have both $a = p$ and $b = q$. (Else the imperfection is trivial.)

Join P_1, P_N to produce the completed net $\mathcal{C}$. Let $\mathcal{Q}$ be the p-net formed from $\mathcal{C}$ by taking P_a, P_b as poles, and omitting one wire joining P_a, P_b. (Of course, there is such a wire; there may be several. It is easy to see, from considerations of "triviality", that $\mathcal{Q}$ is connected, and therefore a p-net.) Applying (2.33) to $\mathcal{C}$, and using (2.35), (2.36), (2.37), we have

$$(6.18) \qquad (H + V) \mid k \cdot (V(ab) - [ab, pq]),$$

where $V(ab)$, $[ab, pq]$ refer to the p-net formed by $\mathcal{C}$ with P_a, P_b as poles, and hence (2.36) refer equally well to $\mathcal{Q}$.

Now, we have $0 < V(ab) - [ab, pq] \leqq$ semiperimeter of $\mathcal{Q}$, with equality only if the current $[ab, qp]$ equals the total current of $\mathcal{Q}$. In this case, $\mathcal{C}$ must consist of two parts, joined only by the two wires P_aP_b and P_pP_q. Further, P_1, P_N, being joined in $\mathcal{C}$ by a wire not P_aP_b or P_pP_q, must lie in the same part. Hence the imperfection in $\mathcal{P}$ with which we started was trivial.

Hence (6.18) gives (since semiperimeter of $\mathcal{Q}$ = complexity of $\mathcal{C} = H + V$)

$$(6.19) \qquad (H + V, k) > 1.$$

Now let n be any positive integer. Join P_1, P_N by $n - 1$ extra wires (of unit conductance). The new p-net will have the same non-trivial imperfection (by (2.36)), so, applying (6.19) to the new net, and using (2.35), (2.32) repeatedly, we have

$$(H + nV, k) > 1.$$

The lemma (6.17) now shows that $(H, V) > 1$. Hence, by (6.15), $\mathcal{P}$ is reducible.

(6.20) COROLLARY. *All irreducible p-nets having no trivial imperfections give perfect squared rectangles.*

(6.21) COROLLARY. *If the complexity of a c-net is prime, all the squared rectangles derived from it (as in* §5.1*) will be perfect.*

These results are sometimes useful as tests for perfection.

For the reduced elements, we can prove (using the Euler polyhedron formula, and some consideration of the various cases)

(6.22) At least three of the reduced elements of any perfect rectangle are even. (Three is the best number possible.)

7. Construction of some special squared rectangles

7.1. **Conformal rectangles.** Two squared rectangles (or p-nets in this plane) which have the same shape (that is, have proportional sides) but are not merely rigid displacements of each other (in the case of p-nets, have not the same normal form) are called *conformal.* An example of a conformal pair is provided by the 9-th order rectangle 64×66 and a 12-th order rectangle of reduced sides 96, 99, whose (reduced) net is specified by: $ga = 31$, $ge = 21$, $gc = 44$, $ea = 10$, $ed = 11$, $ad = 1$, $dc = 12$, $ac = 13$, $ab = 27$, $cb = 14$, $bf = 41$, $cf = 15$.

Two conformal rectangles need not have the same full sides or reduction; for example, the rectangle 96×99 has reduction 3 (cf. (6.21)).

We now show how to construct conformal pairs having the *same* reduced elements (but differently arranged).

Suppose that a p-net $\mathcal{P}$ has a part $\mathcal{Q}$ joined to the rest only at vertices $A_1, \cdots, A_m$, say, and containing no pole different from an A_i. If $\mathcal{Q}$ has rotational symmetry about a vertex P, in which the A's are a set of corresponding points, then a simple symmetry argument shows that the potential of P (in $\mathcal{P}$) will be the mean of the potentials of $A_1, \cdots, A_m$. Hence if this is also true for another vertex P', P and P' will have equal potentials.

Coalesce P and P', forming (if this can be done in the plane) the p-net $\mathcal{P}_2$. If C is the complexity of $\mathcal{P}$, we see from (2.33) that, if $[ab, 1N]$ and $[ab, 1N]_2$ are corresponding elements in $\mathcal{P}$ and $\mathcal{P}_2$ (with 1, N referring to the poles),

$$[ab, 1N]_2 = \frac{V(PP')}{C} [ab, 1N].$$

Hence the elements of $\mathcal{P}_2$ are proportional to those of $\mathcal{P}$; $\mathcal{P}$ and $\mathcal{P}_2$ have the same reduced elements and sides. Their reductions are clearly in the ratio $C: V(PP')$. This construction enables conformal p-nets with the same elements to be written down.

A simple example is shown in Figure 2. Here A_1 and A_2 are poles. The rectangles are perfect and simple, and have reductions 5 and 6, and reduced sides 75 and 112.

In a more complicated example, illustrating a variation on the method, we make the potentials of three points P_1, P_2, P_3 equal. Although the network we start with is not planar, it becomes so when either P_1P_2 or P_1P_3 coincide. Such a network is specified below. It gives conformal simple perfect rectangles of the 28-th order, with reductions 96 and 120, reduced sides 6834 and 14065, and reduced elements: $A_1a = 3288$, $A_1P_1 = 3480$, $A_1b = 2512$, $A_1d = 2247$, $A_1i = 2538$, $aP_3 = 192$, $aA_3 = 3096$, $bP_3 = 968$, $bA_2 = 1544$, $P_1A_2 = 576$, $P_1A_3 = 2904$, $P_3c = 1160$, $A_2c = 584$, $cA_3 = 1744$, $de = 1014$, $dP_2 = 1233$, $eA_2 = 795$,

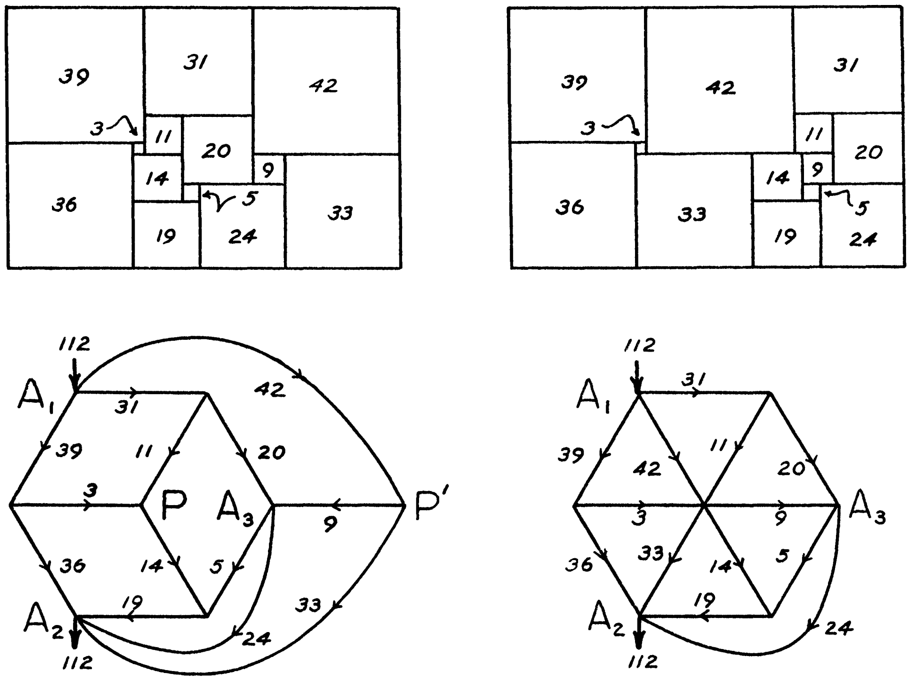

FIG. 2

$eP_2 = 219$, $iP_2 = 942$, $ih = 1596$, $P_2h = 654$, $P_2f = 579$, $P_2g = 1161$, $hA_3 = 2250$, $A_2f = 3$, $fg = 582$, $gA_3 = 1743$, $A_2A_3 = 2328$. (The poles are A_1 and A_3.)

These examples show that, even when the sides and elements of a simple perfect rectangle are given, the configuration is far from uniquely determined.

We now turn to the opposite problem of constructing conformal pairs of squared rectangles having *different* sets of elements. Again, symmetry considerations enable us to do this. We are led to pairs of rectangles (and p-nets) which are not merely conformal but have the same full sides. Such pairs are said to be *equivalent*.

7.2. **Symmetry method.** Let a p-net $\mathcal{P}$ have a part $\mathcal{Q}$ joined to the rest only at vertices $A_1, \cdots, A_m$, and containing no pole different from an A_i. Suppose that $\mathfrak{Q}$ has rotational symmetry in which the A's are a set of corresponding points, and that $\mathfrak{Q}$ is not identical with its mirror-image. $\mathfrak{Q}$ is the *rotor*, and the wires of $\mathcal{P} - \mathfrak{Q}$ form the *stator*. In $\mathcal{P}$, replace $\mathfrak{Q}$ by its mirror-image. It is easy to see that the full currents in the stator will be entirely unaffected, though (in general) the rotor currents will change. (This can be proved, e.g., by induction over the number of wires in the stator, if we use §2.) So we have, in general, a pair of equivalent rectangles, with different (though overlapping) sets of elements.

One of the simplest examples of this method is shown in Figure 3. This gives equivalent simple perfect rectangles of order 16, reduction 5, and reduced sides 671 and 504.[13]

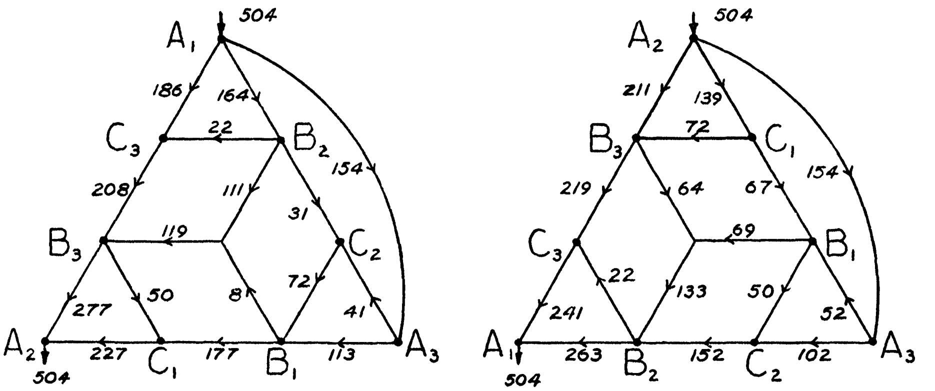

FIG. 3

We may generalize this method by noting that it remains effective when some of the A's are coincided (corresponding to the introduction of "wires of infinite conductance" in the stator). Or, again, we may take the stator to be itself a rotor, with $A_1, \cdots, A_m$ as its set of corresponding points (with possible coincidences). By reflecting both parts we can get pairs of equivalent rectangles having no elements in common.

7.3. **Special methods.** The preceding methods (and similar ones based on duality instead of symmetry) are useful for existence theorems, as in the next section; but other devices are more suitable for producing equivalent rectangles of small orders.

If, in a c-net $\mathfrak{C}$, we can find two wires whose end-points—say P_a, P_b and P_x, P_y, respectively—satisfy

$$V(ab) = V(xy) \qquad \text{(in } \mathfrak{C}\text{)}, \tag{7.31}$$

[13] The rotor of Figure 3 has a remarkable property. If currents I_1, I_2, I_3 (summing to zero) enter the *rotor* (considered as a net) at A_1, A_2, A_3, then the currents in B_3C_1, B_1C_2, B_2C_3 will be $I_1/7$, $I_2/7$, $I_3/7$, respectively. This explains the "extra" equalities of the currents in Figure 3. Other rotors of 15 wires (having the same type of symmetry) behave in a similar way. This phenomenon is not yet fully explained.

then the corresponding p-nets (obtained from $\mathcal{C}$ by omitting each of the two wires in turn, and taking its ends as poles) will be equivalent, if not identical. For they have the same semiperimeter in any case, viz., the complexity of $\mathcal{C}$.

By using the properties of symmetrical or self-dual networks, we can often demonstrate an equality like (7.31). For example, in Figure 4, it is clear that

$$V(gh) = V(cb) \tag{7.32}$$

and

$$[da, gh] = 0. \tag{7.33}$$

Hence (by (7.33) and (2.23) and symmetry)

$$[de, gh] = [ae, gh] = [de, cb]. \tag{7.34}$$

Now, (7.32) and (7.34) imply (if we use (2.37) and (2.33)) that the impedances of gh and cb remain equal when we add a wire joining de. Hence this new c-net

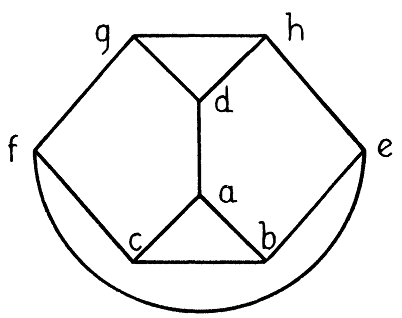

FIG. 4

satisfies (7.31), and so we get a pair of equivalent squared rectangles of the 12-th order. These rectangles are perfect, and provide the simplest example of equivalence among perfect rectangles. They both have reduction 2 and reduced sides 142 and 162. Their (reduced) specifications are respectively:

$gf = 57$, $gd = 85$, $dh = 77$, $de = 12$, $ad = 4$, $fe = 40$, $be = 13$, $eh = 65$, $ab = 3$, $ca = 7$, $cb = 10$, $fc = 17$; and $cf = 59$, $ca = 83$, $fe = 40$, $fg = 19$, $gh = 10$, $he = 11$, $gd = 9$, $dh = 1$, $ad = 4$, $de = 12$, $eb = 63$, $ab = 79$.

8. Construction of perfect squares

8.1. **Definition.** Two conformal rectangles are said to be *totally different* if C_2 times an element of the first is never equal to C_1 times an element of the second, where C_1, C_2 are their respective (corresponding) horizontal sides.

For equivalent rectangles this is equivalent to: No element of the first equals an element of the second.

A pair of totally different simple perfect squared rectangles gives us a perfect square at once; we have only to place them as in Figure 5, and add two corner squares. This idea, though often in modified form, underlies all the constructions for perfect squares in this paper.

(8.11) It is easy to show (by the use of determinants) that if H, V and H', V' are the full sides of the rectangles used in this construction, then the resulting square will have full side $(H + V)\cdot(H' + V')$. In particular, if the rectangles are *equivalent*, the full side of the square is the square of an integer.

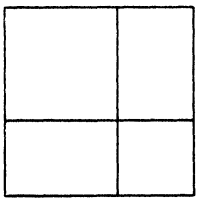

FIG. 5

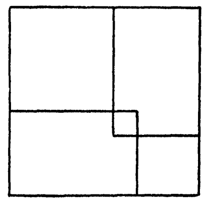

FIG. 6

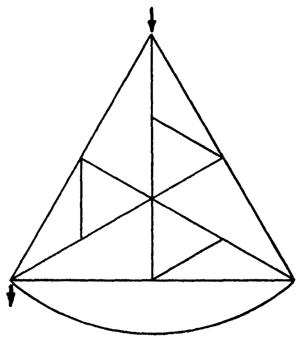

FIG. 7

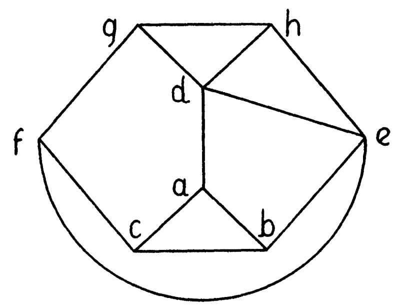

FIG. 8

8.2. **Symmetry method.** Equivalent perfect rectangles constructed as in §7.2 can be used to give us a perfect square. The stator is taken to be a single wire A_iA_j (drawn outside the rotor), one of whose end-points is a pole. The equivalent rectangles so obtained will have, *in general,*[14] just one element in common, the element corresponding to this stator. As this element is placed at a *corner* in both rectangles, we may "overlap" the rectangles as in Figure 6 to get a square.

One of the simplest perfect squares formed in this way is based on the rotor and stator shown in Figure 7. The square is of the 39-th order.

(8.21) It can be shown that, if H, V are the full sides of the equivalent rectangles used in this construction (§8.2), and E is the common element, then the full

[14] The "exceptional case", in which two elements from the following set: the rotor, its reflection, and the stator-element, are equal, seems in practice to be rare. It does occur, however, if the rotor has trivial imperfections, or if it has too much symmetry, or if it has triad symmetry and only 15 wires (cf. the previous footnote).

side of the resulting squared square is $(H + V - E)^2$, the square of an integer. In the case of *triad* symmetry ($m = 3$ in §7.2), we can show that $E \cdot (2H + 2V - E) = HV$, so that the full side of the squared square is, in this case, $H^2 + HV + V^2$.

8.3. **Perfect squares of smaller orders.** A perfect square of much smaller order is given by an elaboration of §7.3. We can show by an argument similar to that in §7.3, but longer, that in the net shown in Figure 8, $V(cf) = V(ge)$.
(We use the facts that, if g and f are coalesced in Figure 8, the net becomes symmetrical and self-dual, and that Figure 8 results from Figure 4 by joining de.) Hence the two p-nets obtained by taking respectively c, f and g, e as poles in Figure 8 are equivalent (for their horizontal sides both equal the complexity). They are in fact perfect and totally different; and, though not both simple (the c, f one being obviously compound), the method of §8.1 is easily modified to give a perfect square, which is drawn in Figure 9. It is of the 26-th order. (The least possible order of a perfect square is unknown.)

We have also constructed, in a similar way, two perfect squares of the 28-th order, each of full side $(1015)^2$ and reduced side 1015.[15]

8.4. **Simple perfect squares.** The perfect squares constructed so far have all been compound. By generalizing the method of §8.2 to certain "squared polygons", we can obtain "simple" perfect squares.

First, let $\mathfrak{N}$ be a net with $A_1, \cdots, A_m$ as the vertices of its "outside" polygon, in order. Consider an electric flow in $\mathfrak{N}$ in which all of $A_1, \cdots, A_m$ are poles—i.e., in which currents I_i (not all zero) enter $\mathfrak{N}$ at A_i ($\sum I_i = 0$). Suppose that $I_i \geqq 0$ if $i > 1$. (This could be weakened; but some restriction on the order of the ingoing and outgoing currents is necessary.) Then the flow in $\mathfrak{N}$ corresponds to a squared polygon, of angles $\frac{1}{2}\pi$ and $\frac{3}{2}\pi$.

Proof. We reduce the number of poles of $\mathfrak{N}$ as follows: Suppose A_i is at potential V_i. Suppose there is more than one i for which $I_i > 0$; let $1'$, $2'$ be the least and second least such i's. If $V_{1'} = V_{2'}$, coalesce $A_{1'}$ and $A_{2'}$ (by joining them by a line outside the polygon $A_1 \cdots A_m$ and shrinking the line to a point); and let current $I_{1'} + I_{2'}$ enter there, the other currents being as before. The currents in $\mathfrak{N}$ will be unaltered, and there is now one fewer positive current entering the network. If $V_{1'} \neq V_{2'}$, we can suppose $V_{1'} > V_{2'}$. Join $A_{1'}$, $A_{2'}$ by a wire of conductance $I_{2'}/(V_{1'} - V_{2'})$ (passing outside the polygon $A_1 \cdots A_m$) and take currents $(I_{1'} + I_{2'})$ at $A_{1'}$, 0 at $A_{2'}$ and I_i at A_i for the other i's. Again, the currents in $\mathfrak{N}$ will be unaltered, and one fewer positive current enters the system. Repeating this process till there is only one positive external current left, we have the flow in $\mathfrak{N}$ "imbedded" in a flow with only two poles; in fact, in a p-net flow (except that some of the extra wires may have conductances different from 1). This corresponds to a "rectangled rectangle" R.

[15] See [16].

Stripping off the elements of R which correspond to the extra wires, we are left with a squared polygon, corresponding to $\mathfrak{N}$.

Since the currents I_i are (apart from sign) at our disposal, the shape of the squared polygon can be controlled. (It has $m - 2$ degrees of freedom.)

Now take for $\mathfrak{N}$ a pure rotor—i.e., a network having skew symmetry; and suppose that the points $A_1, \cdots, A_m$ are a set of corresponding points in $\mathfrak{N}$.

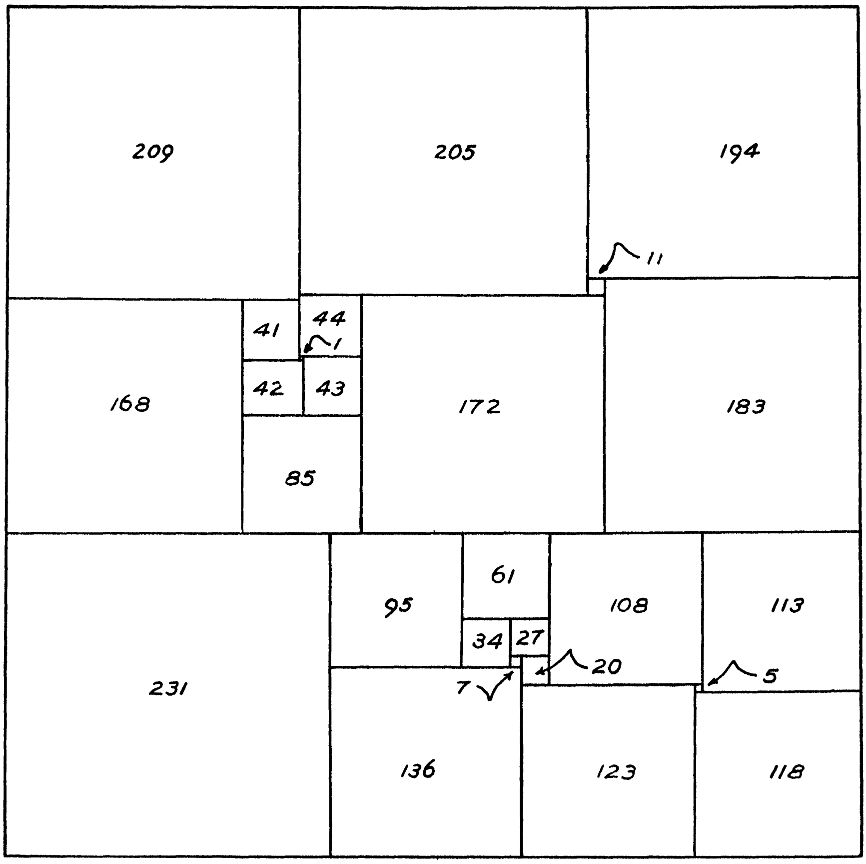

FIG. 9

If $\mathfrak{N}$ is replaced by its reflection (leaving the currents I_i invariant), the new squared polygon will have the same shape as the old—in fact, the two squared polygons will be "equivalent". For, as in §7.2, the rectangled rectangle R will be replaced by an equivalent one, in which the "extra" elements are the same as before.

By combining such a pair of equivalent polygons, as in Figure 10, and arranging their shape so that the overlapped portions coincide with elements (which are then removed), and inserting three extra squares (in the center and at the corners), we can obtain a "simple" perfect square.

For instance, the rotor shown in Figure 11 gives rise to a simple "uncrossed" perfect square of order 55, which, when drawn out, disguises its symmetrical origin very skillfully.

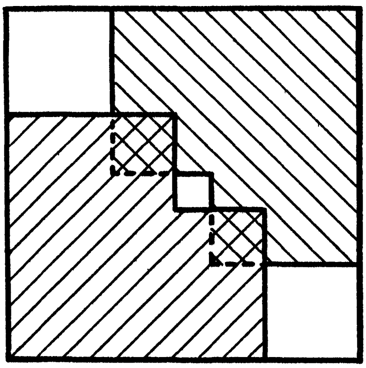

FIG. 10

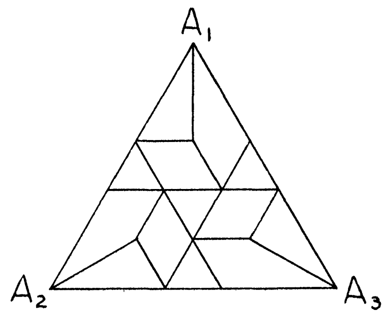

FIG. 11

9. Perfect subdivision of the general rectangle

9.1. We begin by proving:

(9.11) There exist infinitely many totally different perfect squares.

We construct such an aggregate of squares by the method of §8.2, taking for our equivalent rectangles those furnished by the "rotor-stator" diagram (cf. §7.2) of Figure 13. In this diagram, A_1, A_2 are the poles, and the wire A_1A_3 is the stator. The three "resistances" A_1B_2, etc., denote three copies of the p-net of some perfect rectangle. We shall select a sequence $\mathfrak{R}_n$ of suitable p-nets, and, for each $\mathfrak{R}_n$, form the corresponding square $\mathfrak{S}_n$. The sequence $\mathfrak{S}_n$ will then (as follows from (9.39)) have a subsequence of perfect squares, every two of which are totally different. This will prove (9.11).

9.2. **The perfect rectangles $\mathfrak{R}_n$.** Let $\mathfrak{R}_n$ be the p-net shown in Figure 12, with P_0, Q_0 as poles.

Write $\phi_r = [(2 + \sqrt{3})^r - (2 - \sqrt{3})^r]/2\sqrt{3}$. Thus

(9.21) ϕ_r is an integer; $\phi_0 = 0$; and $\phi_{r+1} - 4\phi_r + \phi_{r-1} = 0$.

It will readily be verified that a solution of Kirchhoff's equations is given by:

(9.22) Current in P_0P_r (from P_0 to P_r) is a_r, where

$$a_r = \tfrac{1}{2}\cdot[5\phi_n + \phi_{n-1} + 3\phi_r - 3\phi_{r-1}] \qquad \text{if } 0 < r < n,$$

$$a_{n+1} = 3\phi_n .$$

Current in P_rQ_0 is b_r, where

$$b_r = \tfrac{1}{2}\cdot[5\phi_n + \phi_{n-1} - 3\phi_r - 3\phi_{r-1}] \qquad \text{if } 0 < r < n + 1,$$

$$b_{n+1} = 2\phi_n + \phi_{n-1} .$$

Current in P_rP_{r+1} is c_r, where

$$c_r = 3\phi_r \qquad \text{if } 0 < r < n,$$
$$-c_n = \phi_n - \phi_{n-1}.$$

(This solution is in fact the full flow.)

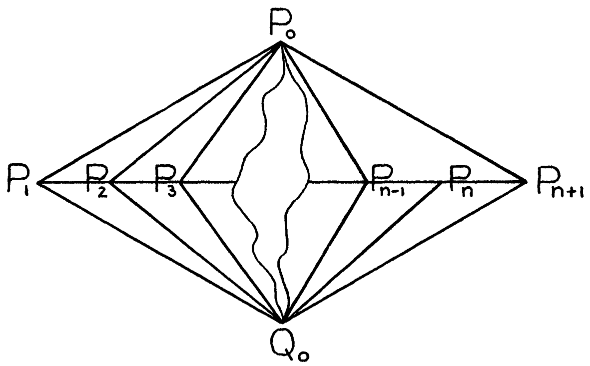

FIG. 12

Also the total current p_n (the horizontal side of $\mathcal{R}_n$), and the total P.D. q_n (the vertical side) are given by:

(9.23) $\quad p_n = \frac{1}{2}\cdot[(5n + 1)\phi_n + (n + 2)\phi_{n-1}]; \qquad q_n = 5\phi_n + \phi_{n-1}.$

Now, if $n > 2$, we see that

$$0 < c_1 < c_2 < \cdots < c_{n-2} < (-c_n) < c_{n-1} < b_n < b_{n+1} < b_{n-1} < b_{n-2}$$
$$< \cdots < b_1 < a_1 < a_2 < \cdots < a_{n-1} < a_{n+1}.$$

Hence

(9.24) $\qquad$ If $n > 2$, $\mathcal{R}_n$ is perfect.

From (9.23), we have

(9.25) $\qquad$ q_n and $p_n/q_n \to \infty$ with n.

For later use, we note that

(9.26) $\qquad (p_n, q_n) \mid 9.$

Proof. From (9.23),

$$(n + 2)q_n - 2p_n = 9\phi_n \qquad \text{and} \qquad (5n + 1)q_n - 10p_n = 9\phi_{n-1}.$$

Now, we can prove by induction (using (9.21)) that $(\phi_n, \phi_{n-1}) = 1$. Thus (9.26) follows.

[(9.24) can be generalized: If in Figure 12 the wire P_0P_n is inserted and the wire P_0P_r removed, where $1 < r \leqq \frac{1}{2}n$, the resulting p-net $\mathcal{R}_{nr}$ is perfect. $\mathcal{R}_{n2}$

is essentially the same as $\mathcal{R}_n$. The reduction ρ_r of $\mathcal{R}_{nr}$ can be calculated; for instance, it can be shown that ρ_r is a factor of $(\phi_r - \phi_{r-1})$; and that $\rho_r = (\phi_r - \phi_{r-1})$ if and only if $n \equiv 0 \pmod{2r - 1}$.]

9.3. We next prove

(9.31) THEOREM. *For all large n, the squared square $\mathcal{S}_n$ is perfect.*

Consider the equivalent p-nets of Figure 13, where each "resistance" denotes a certain p-net $\mathcal{R}$, of horizontal side p and vertical side q. (The other wires have conductance 1, as usual. Later, $\mathcal{R}_n$ will be taken as $\mathcal{R}$.)

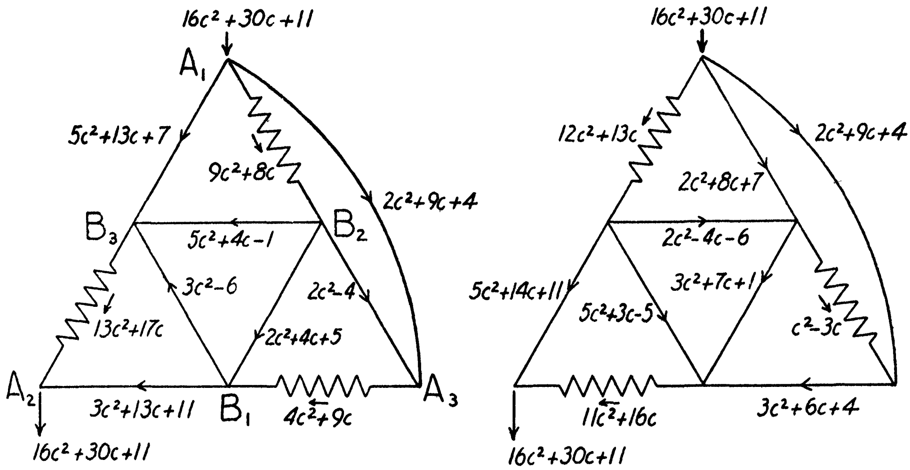

FIG. 13

Setting $c = p/q$ = effective conductance of these resistors, we find that the flows are as indicated in the diagram. (The quantities shown are currents.)

Hence, multiplying through by q^2, and adjoining the extra elements required in forming the squared square $\mathcal{S}$ as in §8.2, we find that the elements of $\mathcal{S}$ (some integral multiple of the reduced elements) are:

$$(9.32)\quad \begin{cases} \text{(A)} & 14p^2 + 21pq + 7q^2,\ \ 5p^2 + 14pq + 11q^2,\ \ 5p^2 + 13pq + 7q^2, \\ & \qquad 5p^2 + 4pq - q^2,\ \ 5p^2 + 3pq - 5q^2, \\ & 3p^2 + 17pq + 20q^2,\ \ 3p^2 + 13pq + 11q^2,\ \ 3p^2 + 7pq + q^2, \\ & \qquad 3p^2 + 6pq + 4q^2,\ \ 3p^2 - 6q^2, \\ & 2p^2 + 9pq + 4q^2,\ \ 2p^2 + 8pq + 7q^2,\ \ 2p^2 + 4pq + 5q^2, \\ & \qquad 2p^2 - 4q^2,\ \ 2p^2 - 4pq - 6q^2. \\ \text{(B)} & \text{Multiples of the elements of } \mathcal{R}\text{, the multipliers being respectively} \\ & 13p + 17q,\ 12p + 13q,\ 11p + 16q,\ 9p + 8q,\ 4p + 9q,\ p - 3q. \end{cases}$$

We also find that

(9.33) The side of $\mathfrak{S}$ is $19p^2 + 47pq + 31q^2$.

Now take $\mathfrak{R}$ to be $\mathfrak{R}_n$, so that $p = p_n$ and $q = q_n$; and let n be so large that (in virtue of (9.25))

$$p_n > 180q_n. \tag{9.34}$$

We prove that, under this condition, $\mathfrak{S} = \mathfrak{S}_n$ is perfect.

(9.35) *The elements* (A) *are all different, and no element* (A) *equals an element* (B).

For the elements (A) in the above list are in strictly decreasing order; so no two of them are equal. Also the least element (A) is $2p^2 - 4pq - 6q^2$ which $> (13p + 17q)q$, which is greater than any element (B). Thus (9.35) follows.

(9.36) *No two elements* (B) *are equal.*

For suppose that two such elements are equal:

$$\xi(\alpha p + \beta q) = \eta(\gamma p + \delta q), \tag{9.37}$$

where ξ, η are elements of $\mathfrak{R}_n$, and $\alpha p + \beta q$, $\gamma p + \delta q$ are two multipliers of (9.32). They are *different* multipliers; for $\mathfrak{R}_n$ is perfect by (9.24). Hence, by inspection of (9.32), $\alpha\delta - \beta\gamma \neq 0$. But, from (9.37), $(\alpha\xi - \gamma\eta)p = (\delta\eta - \beta\xi)q$. Hence

$$p \mid (\delta\eta - \beta\xi) \cdot (p, q). \tag{9.38}$$

Now, if $\delta\eta - \beta\xi = 0$, we have (since $p \neq 0$) $\alpha\xi - \gamma\eta = 0$, and hence, if we eliminate ξ, η (which are not zero), it follows that $\alpha\delta - \beta\gamma = 0$. So we have $0 < |\delta\eta - \beta\xi| < 20q$ (by inspection of (9.32), since $0 < \xi, \eta < q$).

Hence, if we use (9.26), (9.38) gives $p < 180q$. This contradicts (9.34). And (9.31) now follows from (9.35) and (9.36); the squares $\mathfrak{S}_n$ are perfect, for large enough n.

(9.39) THEOREM. *Given any large enough n, then for all large enough N, $\mathfrak{S}_n$ and $\mathfrak{S}_N$ are totally different.*

Write $p_n = p$, $q_n = q$, $p_N = P$, $q_N = Q$. We bring $\mathfrak{S}_n$ and $\mathfrak{S}_N$ to the same size by multiplying the elements of $\mathfrak{S}_n$ (as given by (9.32)) by $19P^2 + 47PQ + 31Q^2$ and those of $\mathfrak{S}_N$ by $19p^2 + 47pq + 31q^2$. (This follows from (9.33).)

(9.40) *Each element* (B) *of $\mathfrak{S}_N$ is less than every element of $\mathfrak{S}_n$.*

For a typical element (B) of $\mathfrak{S}_N$ is

$$e = (\alpha P + \beta Q) \cdot (19p^2 + 47pq + 31q^2), \qquad \text{where } |\alpha|, |\beta| \leqq 17.$$

If n and N are large, this gives $e < 360Pp^2$. (This follows from (9.25).) But each element of $\mathfrak{S}_n$ is at least as large as P^2p (times some non-zero constant). Hence if $n >$ some n_0, and if then $N >$ some $N_0(n)$, so that P is large compared with p (see (9.25)), we have $e <$ each element of $\mathfrak{S}_n$.

(9.41) *Each element* (A) *of* $\mathfrak{S}_N$ *is greater than every element* (B) *of* $\mathfrak{S}_n$.

For any element (A) of $\mathfrak{S}_N$ is at least as large as P^2p^2 (times some non-zero constant), whereas an element (B) of $\mathfrak{S}_n$ is less than $360P^2p$.

(9.42) *No element* (A) *of* $\mathfrak{S}_N$ *can equal any element* (A) *of* $\mathfrak{S}_n$.

Otherwise we have

$$(aP^2 + bPQ + cQ^2)\cdot(19p^2 + 47pq + 31q^2)$$
$$= (a'p^2 + b'pq + c'q^2)\cdot(19P^2 + 47PQ + 31Q^2),$$

where by (9.32) a, a', etc., are integers numerically less than 22. Hence

$$P^2\cdot[(19a - 19a')p^2 + (47a - 19b')pq + (31a - 19c')q^2] = \text{similar terms in } PQ \text{ and } Q^2. \tag{9.43}$$

Now, $47a - 19b' \neq 0$; for otherwise $19 \mid a$, whereas $0 < a < 19$ (from (9.32)). Hence the left side of (9.43) is numerically at least as large as P^2pq (times some non-zero constant); in fact, if $a \neq a'$, it is as large as P^2p^2. But the right side of (9.43) is at most PQp^2 (times a constant). Hence, if N is taken large enough, so that P dominates both p and Q (this is possible, by (9.25)), (9.43) is impossible.

(9.40), (9.41), and (9.42) imply (9.39).

(9.44) COROLLARY. *There is a sequence* $\{\mathfrak{T}_n\}$ *of perfect squares, every two of which are totally different.*

This is immediate from (9.31) and (9.39) and proves (9.11).

A rough calculation shows that we may take $\mathfrak{T}_r = \mathfrak{S}_{103(r+1)}$. This could probably be greatly improved.

(9.45) THEOREM. *Any rectangle whose sides are commensurable can be squared perfectly in an infinity of totally different ways.*

Magnifying the rectangle suitably, we may suppose that its sides are integers h, k. Divide it into hk squares of side 1, by lines parallel to its sides. Take any positive integer n, and replace the i-th of these unit squares by $\mathfrak{T}_{nhk+i}$ (suitably contracted). By (9.44), this gives, for each n, a perfect subdivision of the given rectangle; and these subdivisions for any two values of n are "totally different".

Using the theorem of (2.14), we see that a rectangle can be squared *perfectly* if it can be squared at all.

It is plausible that any commensurable-sided rectangle can be squared perfectly and *simply*; possibly this can be proved in a similar way if we use some extension of §8.4; but this seems to involve laborious calculations.

10. Some generalizations

We mention briefly some of the extensions of the methods and results of this paper. A fuller discussion may perhaps appear later.

10.1. **Rectangled rectangles.** An immediate and natural generalization (as pointed out in §1.2) is to the problem of a rectangle dissected into a finite number of rectangles. The wires of the p-net merely have general (not necessarily equal) conductances.

There is also (cf. §8.4) a rather trivial extension in which the dissection is of a polygon (of angles $\frac{1}{2}\pi$ and $\frac{3}{2}\pi$). A more natural generalization, however, is given in the following section.

10.2. **Squared cylinders and tori.** We may regard a squared rectangle, after identification of its left and right sides, as a "trivial" example of a squared cylinder. The squared cylinders are found to correspond exactly to the relaxation of the condition (1.12) that no circuit of the p-net may enclose a pole. A second step brings us to the "squared torus". Using the existence theorem of (9.45), we can easily construct such figures. It is also possible to construct a *simple* non-trivial perfect torus; but this is not so easy.

Of course, the word "squared" may be replaced by "rectangled".

10.3. **Triangulations of a triangle.** In a rather different direction, we may consider dissections of a triangle into a finite number of triangles; particularly when all the triangles considered are *equilateral.* It is easily proved that *there is no perfect equilateral triangle*; i.e., that in any such dissection of an equilateral triangle into equilateral triangles, two of the latter are equal. Apart from this, the theory extends fairly completely. Duality relations, for example, are replaced by "triality" relations. We could also consider dissections into a mixture of equilateral triangles and regular hexagons, no two of these elements having equal sides; essentially this amounts to agglomerating the imperfections of an "equilateral triangled triangle" together by sixes. There is no difficulty in constructing such figures empirically, or in finding "perfect isosceles right-angled triangles"; however, it can be done by using the theory.

10.4. **Three dimensions.** We have seen that the "p-net" and its generalizations are satisfactory for plane dissections. As yet, however, there is no satisfactory analogue in three dimensions. The problem is less urgent, because *there is no perfect cube* (*or parallelopiped*). That is, in any dissection of a rectangular parallelopiped into a finite number of cubes ("elements"), two of the latter are equal.

Proof. It is easily seen that in any perfect rectangle, the smallest element is not on the boundary of the rectangle. Suppose we have a "perfect" cubed parallelopiped P. Let R_1 be its base. The elements of P which rest on R_1 "induce" a dissection of R_1 into a perfect rectangle. (We can clearly assume that more than one cube rests on R_1.) Let s_1 be the smallest element of R_1. Let c_1 be the corresponding element of P. Then c_1 is surrounded by *larger*, and therefore *higher*, cubes on all four sides; for, as remarked above, s_1 is surrounded by larger squares. Hence the upper face of c_1 is divided into a perfect rectangle R_2 by the elements of P which rest on it; let s_2 be the smallest element of R_2; and so on. In this way, we get an infinite sequence of elements c_n of P, all different (for $c_{n+1} < c_n$). This is a contradiction.

This proof excludes generalizations of "perfect cylinders" to three (or more, a fortiori) dimensions; but it does not exclude the possibility of a *perfect three-dimensional torus* (product of three circles). It is not known whether such a thing can exist.

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