THE DISSECTION OF EQUILATERAL TRIANGLES INTO EQUILATERAL TRIANGLES

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1. Introduction

In a previous joint paper ('The dissection of rectangles into squares', by R. L. Brooks, C. A. B. Smith, A. H. Stone and W. T. Tutte, *Duke Math. J.* 7 (1940), 312–40), hereafter referred to as (A) for brevity, it was shown that it is possible to dissect a square into smaller unequal squares in an infinite number of ways. The basis of the theory was the association with any rectangle or square dissected into squares of an electrical network obeying Kirchhoff's laws. The present paper is concerned with the similar problem of dissecting a figure into equilateral triangles. We make use of an analogue of the electrical network in which the 'currents' obey laws similar to but not identical with those of Kirchhoff. As a generalization of topological duality in the sphere we find that these networks occur in triplets of 'trial networks' N^1, N^2, N^3. We find that it is impossible to dissect a triangle into unequal equilateral triangles but that a dissection is possible into triangles and rhombuses so that no two of these figures have equal sides. Most of the theorems of paper (A) are special cases of those proved below.

We define a *triangulation of order n* of any region and in particular of an equilateral triangle Δ as a dissection of the region into $n > 1$ closed equilateral triangles, $E_1, E_2, \ldots, E_n$, called the *elements* of the triangulation, which between them completely fill the region and which do not overlap except at their boundaries. It is evident that in any such dissection of Δ the elements fall into two mutually exclusive classes, those placed similarly to Δ which will be called *positive* elements, and those placed similarly to the triangle formed by rotating Δ through an angle π which will be called *negative* elements. The *size* x_r of the element E_r is defined to be the length of the side of E_r taken with a positive or negative sign according as E_r is a positive or a negative element. The triangulation of Δ is called *perfect* if no two elements have the same size. Thus a perfect triangulation has at most two elements with a given length of side, and if it has two then one is a positive and the other a negative element.

In § 2 we describe a graphical representation $M(T)$ of a triangulation T of Δ. We call $M(T)$ the *bicubical map* of T. From $M(T)$ we get three networks $N^1(T)$, $N^2(T)$, $N^3(T)$. We show that with each of these three networks there is associated a set of equations, analogous to Kirchhoff's laws, connecting the sizes of the elements of T. One result of this section is that in any triangulation of an equilateral triangle Δ there must be two elements with a common side, and therefore with equal and opposite sizes. In § 3 we develop the theory of these equations and show that the sizes x_r of the elements E_r must be commensurable.

In § 4 we give an independent definition of a bicubical map, and show that from any such bicubical map we can derive a triangulation of an equilateral triangle.

In § 5 we discuss triangulations of parallelograms, including as a special case squared rectangles ((A), Introduction). § 6 generalizes the main duality theorem (3·25) of (A).

Most of the theorems proved below were discovered, or at least conjectured, during the researches which led to the results of paper (A). Here they have been systematized and missing proofs, such as that of § 6, supplied. I am indebted to the other authors of (A) for permission to use joint results and for helpful criticism during the preparation of the present paper. In particular I have to thank Dr C. A. B. Smith for the conception of the bicubical map of a triangulation, and the definitions of a 'perfect triangulation' and a 'standardized matrix'.

2. Triangulations of triangles

2·1. Consider a triangulation T of order n of an equilateral triangle Δ with elements $E_1, E_2, \ldots, E_n$.

Let the sides of Δ be S_1, S_2 and S_3. Clearly any side of an element of T is parallel to just one of these.

Let v be any vertex of an element E_r of Δ, but not a vertex of Δ. Since E_r occupies an angle $\frac{1}{3}\pi$ at the vertex v we have two possibilities. Either six distinct elements have a vertex at v (when v will be called a *cross*), or just three distinct elements have a vertex at v, the remaining angle of π at v being occupied by another element or by the exterior of Δ.

2·2. Theorem. *For any triangulation T of an equilateral triangle Δ, we can find closed straight segments p_i^σ ($\sigma = 1, 2, 3$; $i = 1, 2, \ldots, m_\sigma$), where m_1, m_2, m_3 are some positive integers, such that* (i) *the union of the p_i^σ is the union of the sides of the E_r, each side of each E_r being contained in some p_i^σ,* (ii) *p_i^σ is parallel to the side S_σ of Δ,* (iii) *two distinct segments p_i^σ have at most one point in common, and* (iv) *if v is a vertex of E_r and not a vertex of Δ, then v is an interior point of just one of the segments p_i^σ.* (An *interior point* of p_i^σ is a point contained in, and not an end-point of, p_i^σ.)

To prove this we consider the union U_σ of those sides of elements of T which are parallel to S_σ. Its connected components are closed straight segments. Let us try taking these as the segments p_i^σ for each value of σ. The segments p_i^σ thus defined evidently satisfy conditions (i) to (iv) save only that (iv) fails at each cross; a cross is an interior point of just three of these segments. Clearly, by subdividing at each cross any two of the segments intersecting there we obtain a set of segments satisfying the conditions of the theorem.

2·3. Suppose that we have a set of segments p_i^σ satisfying the conditions of § 2·2. Then if v is a vertex of an element of T but not a vertex of Δ, there is a unique segment p_i^σ having v as an interior point (§ 2·2 (iv)). We call this the *bisector* $B(v)$ of v. It defines two angles of π at v, and each of these angles contains either 0 or 3 elements of T which have v as a vertex. We call a set of three elements occupying either of these angles a *triplet* of T with *focus* v. The number of triplets of which v is a focus is thus two if v is a cross, and one otherwise. It is convenient to extend this definition by calling the set of three elements of T which meet vertices of Δ a triplet (the *exterior* triplet of Δ).

Δ is regarded as imbedded in the closed plane Z^2, and the point J at infinity is called the focus of this special triplet.

We suppose the triplets enumerated as $F_0, F_1, \ldots, F_{q-1}$, the exterior triplet being F_0.

With these definitions it is clear that each element of T belongs to just three triplets, one for each vertex.

We can represent the relations between the elements and triplets of T by a linear graph $G(T)$. $G(T)$ has $n+q$ vertices, $e_1, e_2, \ldots, e_n$ corresponding to the elements, and $f_0, f_1, \ldots, f_{q-1}$ corresponding to the triplets. Two vertices are joined by at most one edge, and each edge has one end an e_r and the other an f_s. There is an edge joining e_r and f_s if and only if E_r belongs to the triplet F_s.

We see that each vertex of $G(T)$ is incident with just three edges. This property is expressed by saying that $G(T)$ is *cubical*. Moreover, the vertices of $G(T)$ fall into two exhaustive and mutually exclusive classes $\mathbf{E}$ and $\mathbf{F}$—the set of the e_r and the set of the f_s respectively—such that each edge is incident with one member of each class. We express both properties by saying that $G(T)$ is *bicubical*.

Since each edge joins two vertices, one of class $\mathbf{E}$ and the other of class $\mathbf{F}$, and each vertex is incident with just three edges, we see that n, the number of members of $\mathbf{E}$ must equal q, the number of members of $\mathbf{F}$.

$$n = q. \tag{1}$$

2·4. THEOREM. *$G(T)$ can be realized in the closed plane.*

Take e_r to be the centre of E_r. Make straight joins from e_r to each of the vertices of E_r (for each r). A straight join to a vertex of Δ is to be continued through that vertex to the point J at infinity.

By this construction each element, and also the exterior of Δ, is divided into three 3-sided regions. We call these regions *subelements* of T.

Consider the linear graph whose edges are these straight joins and whose vertices are the e_r and the foci of the triplets. This would be a realization of $G(T)$ in Z^2 were it not that each cross X is the focus of two triplets. As the two triplets are separated at X by the bisector $B(X)$ we can 'pull apart' the representative points of the two triplets at each cross and so obtain a realization of $G(T)$ in Z^2. More precisely we choose some positive ϵ less than half the length of the side of the smallest element of T, and at each cross X replace the part of the linear graph within ϵ of X by the two arcs of the circle of radius ϵ and centre X for which the radius makes an angle not less than $\frac{1}{6}\pi$ with the bisector of X. The midpoints of these two circular arcs are taken as the representative points f_s corresponding to the two triplets.

2·5. Let P_i^q be a closed polygon constructed from the union of those (closed) subelements of T which have a side in p_i^q by adding every point of Z^2 distant not more than ϵ from any cross which is an interior point of p_i^q and subtracting every point which is distant less than ϵ from any cross which is an end-point of p_i^q.

It is easily verified that P_i^q is simply connected (its boundary being a simple closed curve), that no two of the P_i^q have an interior point in common, and that the realization of $G(T)$ obtained in § 2·4 contains the boundary, but no interior point of each P_i^q.

the polygons P_i^σ and whose 1-section (linear graph constituted by its edges and vertices) is our realization of $G(T)$. We call this 2-complex the *bicubical map* $M(T)$ of T.

2·6. The index σ will be called the *colour* of P_i^σ. Since E_r has a side parallel to each S_σ it follows from § 2·6 that e_r is incident with just one P_i^σ of each colour σ. Consequently each edge of $M(T)$ is incident with two 2-cells of different colours. For each edge is incident with a member of **E**. The remaining colour will be called the *colour of the edge*. Since each edge is incident with a member of **F** it follows further that the three 2-cells incident with any member of **F** have three different colours. We denote by $K(\sigma)$ the class of the 2-cells of colour σ, and by $L(\sigma)$ the class of the edges of colour σ. By the above considerations the three classes $K(\sigma)$ are exhaustive and mutually exclusive, and so are the three classes $L(\sigma)$.

2·7. By the construction of § 2·5 it follows that e_r is incident with P_i^σ if and only if E_r has a side contained in p_i^σ. Also f_s is incident with P_i^σ if and only if the focus of F_s is in p_i^σ and also p_i^σ contains a side of some member of F_s, save only that f_0 is incident with each 2-cell of M corresponding to a side S_σ of Δ. Henceforth we shall assume that S_σ is p_1^σ.

We denote by W_σ the vertex of Δ opposite S_σ.

THEOREM. *Each (closed) edge of $M(T)$ meets the boundaries of just four 2-cells.*

Let L be a 1-cell of $M(T)$ with end-points e_r, f_s, and colour σ. Let P_j^σ, P_k^σ be the 2-cells of colour σ incident with e_r and f_s respectively. Then they are distinct, for the element E_r (by the above considerations) has one side in p_j^σ and the opposite vertex in p_k^σ except when p_k^σ is S_σ and E_r has a vertex at W_σ.†

So, besides the boundaries of its two incident 2-cells, L meets the boundaries of just two other 2-cells.

2·8. We suppose henceforth that the edges of $M(T)$ are oriented, with positive ends e_r and negative ends f_s. We say that an edge is *directed from* its positive end, and *to* its negative end.

2·9. For each i $(1 \leqslant i \leqslant m_1)$ let us identify all the points of the closed 2-cell P_i^1.

This process does not identify the end-points of any edge of $M(T)$ not incident with a member of $K(1)$. For by § 2·7 the positive end of such an edge would represent an element of T having one side and also the opposite vertex in the same segment p_j^1, or else the edge would represent an element of T with one side in S_1 and having the opposite vertex of Δ as a vertex, which is impossible since $n > 1$.

The result of the identifications is thus clearly a cellular 2-complex $N^1(T)$ which is a dissection of a space homoeomorphic to Z^2. Its vertices are the P_i^1. Its 2-cells are the P_j^2 and the P_k^3. Its edges are the edges of $L(1)$. The edges and 2-cells have the same mutual incidence relations as they have in $M(T)$, and any one of them is incident with P_i^1 if and only if it is incident in $M(T)$ with an edge or vertex incident with P_i^1.

In a similar way, by operating on the 2-cells P_j^2 or P_k^3 instead of the P_i^1 we obtain 2-complexes $N^2(T)$ and $N^3(T)$ respectively. We say that the three $N^\sigma(T)$ constitute

† In the latter case p_j^σ and p_k^σ are distinct because, since $n > 1$, E_r is not the whole of Δ.

a set of *trial* 2-complexes provided that their edges are oriented according to the following rule; *the positive (negative) end of an edge L of $N^\sigma(T)$, when regarded as a closed 2-cell of $M(T)$, contains the positive (negative) end of the edge in $M(T)$.*

We shall see later that triality can be regarded as a generalization of topological duality in the 2-sphere.

We have seen in § 2·7 that in $M(T)$ the vertex e_r is incident with just one edge of each colour. We shall denote the edge of colour σ incident with e_r by L_r^σ. Of the three edges incident with e_r only L_r^σ is an edge of $N^\sigma(T)$. Thus to each element E_r of T there corresponds a unique edge L_r^σ of $N^\sigma(T)$.

The edges of $N^\sigma(T)$ incident with the vertex P_i^σ, taken in their cyclic order at P_i^σ, are directed alternately to and from P_i^σ. This follows from the fact that members of **E** and **F** must alternate in the boundary of the 2-cell P_i^σ of $M(T)$, since $G(T)$ is bicubical (§ 2·3).

2·10. We define a matrix $\{c_{rs}^\sigma\}$ as follows:

If $r \neq s$, then $-c_{rs}^\sigma$ is the number of edges of $N^\sigma(T)$ directed from P_r^σ to P_s^σ, and c_{rr}^σ is the number of edges of $N^\sigma(T)$ directed from P_r^σ†. We note that

$$\sum_s c_{rs}^\sigma = 0 \quad \text{and} \quad \sum_r c_{rs}^\sigma = 0. \tag{2}$$

The first of these follows immediately from the definition of c_{rs}^σ. For the proof of the second we require also the result that the total number of edges directed to a given vertex of $N^\sigma(T)$ is equal to the total number directed from that vertex (since edges of the two kinds alternate at the vertex).

2·11. Let W_σ be the vertex of Δ opposite S_σ. We can suppose that the element of T which meets W_σ is E_σ. Let Σ_σ denote the side of E_σ opposite W_σ.

It is evident that if p_i^σ is not S_σ, the sum of the x_r (see Introduction) taken over all E_r having a side in p_i^σ is zero. But if p_i^σ is S_σ the sum is the length X of the side of Δ.

Putting this in terms of $M(T)$ we find that the sum of the x_r taken over all e_r incident with a given P_i^σ is 0 or X according as P_i^σ is not or is incident with the special vertex f_0.

Let V_r^σ denote $2/\sqrt{3}$ times the distance of p_r^σ from S_σ measured positively towards W_σ. Then if E_h has a side in p_r^σ and the opposite vertex in p_s^σ we have

$$x_h = V_s^\sigma - V_r^\sigma. \tag{3}$$

This equation applies for each E_h except E_σ (W_σ is the only vertex of an element of T not in a p_i^σ, for a fixed σ).

Using the above result for $M(T)$, and (2) and (3), we find that

$$\sum_s c_{rs}^\sigma (V_s^\sigma - V_r^\sigma) = \sum_s c_{rs}^\sigma V_s^\sigma = \begin{cases} 0 & (\text{if } p_r^\sigma \text{ is not } S_\sigma), \\ -X & (\text{if } p_r^\sigma \text{ is } S_\sigma), \end{cases} \tag{4}$$

provided that p_r^σ is not Σ_σ^1.

If p_r^σ is Σ_σ we find by analogous considerations that

$$\sum_s c_{rs}^\sigma V_s^\sigma = x_\sigma - (0 - (X - x_\sigma)) = X. \tag{5}$$

† For $r \neq s$, c_{rs} is thus minus the number of elements of T with bases on p_r^σ and vertices on p_s^σ (except that for the purposes of this enumeration the element having a vertex at W_σ is deemed to have it on S_σ).

Equations (4) and (5) constitute the set of linear equations associated with $N^{\sigma}(T)$ which is referred to in the introduction. It will be shown in the next section that when X is given they uniquely determine the differences of the V_r^{σ} (for any fixed σ) and so also the x_r.

2·12. THEOREM. *In any triangulation T of a triangle Δ some two elements have a side in common.*

Let α_0, α_1 and α_2 denote the number of vertices, edges and 2-cells of $M(T)$ respectively. By § 2·3 we have $\alpha_0 = 2n$ and $\alpha_1 = 3n$. Hence by the Euler polyhedron formula it follows that $\alpha_2 = n+2$.

Let c_m be the number of 2-cells having m sides. Since members of **E** and members of **F** alternate in the boundary of any 2-cell of $M(T)$, c_m vanishes for all odd m. Hence

$$n+2 = c_2+c_4+c_6+\ldots = \alpha_2$$

and

$$3n = \tfrac{1}{2}(2c_2+4c_4+6c_6+\ldots) = \alpha_1.$$

Hence

$$6 = 2c_2+c_4-c_8-2c_{10}-3c_{12}-\ldots. \tag{6}$$

But by the theorem of § 2·7, $c_2 = 0$, for a side of a 2-sided 2-cell could not satisfy that theorem. Hence by (6), $c_4 \geqslant 6$. Since f_0 is incident with just three 2-cells of $M(T)$, it follows that there is a 2-cell P_i^{σ} of $M(T)$ not incident with f_0 and having just four sides. Then P_i^{σ} is incident with just two of the e_r. Since p_i^{σ} is not S_{σ} (P_i^{σ} is not incident with f_0) it follows that p_i^{σ} is a side of each of the two corresponding elements E_r.

3. THE METRICAL PROPERTIES OF TRIANGULATIONS

3·1. Let N be an oriented network such that each edge is incident with two distinct vertices. We suppose that with each edge there is associated a real number called the *conductance* of the edge.

We suppose the vertices of N to be p in number, and enumerate them as $P_1, P_2, \ldots, P_p$.

Let $-c_{rs}$ $(r \neq s)$ be the sum of the conductances of all the edges which are directed from P_r and to P_s, and let c_{rr} be the sum of the conductances of the edges directed from P_r. Clearly

$$\sum_s c_{rs} = 0. \tag{7}$$

From (7) we can readily show that the cofactors of the elements of any particular row of the matrix $\{c_{rs}\}$ are all equal. We call their common value for the rth row the *complexity of N at P_r* and denote it by $C_r(N)$ or simply by C_r.

When N has only one vertex P_1 we write $C_1(N) = 1$.

If it is also true that

$$\sum_r c_{rs} = 0, \tag{8}$$

we can likewise deduce that the cofactors of the elements of any particular column of $\{c_{rs}\}$ are all equal, whence it follows that C_r has the same value $C(N) = C$ say for each r. We then call $C(N)$ the *complexity* of N.

Equation (8) is not true in general. We note, however, that it is true for the matrix obtained from $\{c_{rs}\}$ by multiplying the elements of each row by the corresponding C_r. The sum of the elements of any column of this matrix is equal to the determinant of $\{c_{rs}\}$ which, by (7), is 0. We call this matrix the *standardized matrix of N*.

3·2. The second cofactor obtained by taking the cofactor of c_{su} in the cofactor of c_{rt} in $\{c_{rs}\}$ (for $r \neq s$, $t \neq u$) is denoted by $(rs\,.\,tu)$. We also write

$$(rr\,.\,tu) = (rs\,.\,tt) = 0 \quad (\text{all } r, s, t, u). \tag{9}$$

From this definition we have

$$(rs\,.\,tu) = -(sr\,.\,tu) = -(rs\,.\,ut). \tag{10}$$

3·3. Consider the linear equations

$$\sum_u c_{tu} V_u = \delta_{tr} H_r - \delta_{ts} H_s \tag{11}$$

in the unknowns V_u ($\delta_{ii} = 1$, $\delta_{ij} = 0$ if $i \neq j$). We suppose $r \neq s$. With respect to this set of equations we call P_r the *positive* and P_s the *negative pole* of N. A necessary and sufficient condition for the consistency of equations (11) is that $\{c_{rs}\}$ and the augmented matrix formed by adding to it a column whose tth element is $\delta_{tr} H_r - \delta_{ts} H_s$ shall have the same rank†. For this it is necessary that the determinant of each square submatrix of order p of the augmented matrix shall vanish, i.e. that

$$H_r C_r = H_s C_s. \tag{12}$$

If (12) is true, and if also $C_s \neq 0$, the equations will be consistent, $\{c_{rs}\}$ and the augmented matrix having the same rank $p-1$. If this is the case we can ignore the sth equation, which will be dependent upon the others. Multiplying each of equations (11) other than the sth by -1 and adding to equation (7) multiplied by an arbitrarily fixed V_t, we obtain a set of $p-1$ independent linear equations in the $p-1$ unknowns $V_t - V_u$, where t is fixed and $u \neq t$. The determinant D of this set of equations is the complementary minor of c_{st}, whose value is $(-1)^{s+t} C_s \neq 0$. It follows that the $p-1$ equations have a unique solution. In this solution $V_t - V_u$ is the cofactor of c_{ru} in the determinant D, multiplied by $-H_r$ and divided by $(-1)^{s+t} C_s$. That is

$$V_t - V_u = \frac{-H_r}{C_s}(sr\,.\,tu) = \frac{H_r}{C_s}(rs\,.\,tu), \tag{13}$$

by (10).

From (13) we deduce the following polynomial identity (in the variables c_{ij}):

$$(rs\,.\,tu) + (rs\,.\,uv) = (rs\,.\,tv). \tag{14}$$

From the analogous result for the transpose of the standardized matrix of N we have also a polynomial identity

$$C_s(qr\,.\,tu) + C_q(rs\,.\,tu) = C_r(qs\,.\,tu). \tag{15}$$

It is of interest to compare these results with those of § 2·2 of (A). The fundamental distinction is that in (A) the matrix $\{c_{rs}\}$ is symmetrical. Because of this we have for (A) the result $[rs\,.\,tu] = [tu\,.\,rs]$, but in the present theory it is not in general true that $(rs\,.\,tu) = (tu\,.\,rs)$. The theory reduces to that of (A) when we postulate that $\{c_{rs}\}$ is symmetrical, and that two oriented edges of the same conductance c and with the same end-points but with opposite orientations are equivalent to a 'wire' of conductance c. The interpretation of the complexity of an electrical network in terms of subtrees ((A), §(3·1)) also has a simple generalization in the present theory, as we now proceed to show.

† A. C. Aitken, *Determinants and matrices* (Edinburgh, 1939), pp. 69–71.

3·4. As in (A) we define a *subnetwork* of N as a network consisting of all the vertices and some subset of the edges of N. A *subtree* of N is a subnetwork which is a tree, i.e. which is connected and which contains no simple closed curve. If the number of edges of the subtree T of N which have the vertex P_r as positive (negative) end is 0 for a particular value k of r and 1 for every other value of r, then T is said to *converge to* (*diverge from*) P_k.

We enumerate the subtrees of N which converge to the vertex P_k, and denote by Π_j the product of the conductances of the edges of the jth of them. We write

$$U_k(N) = \sum_j \Pi_j. \tag{16}$$

3·5. Suppose that N has at least one edge and at least three vertices. Let P_j and P_k be the positive and negative ends respectively of some edge L, of conductance c. Let N' be derived from N by suppressing L and let N'' be derived from N by suppressing all edges joining P_j and P_k and then identifying P_j and P_k. From the definition of $C_k(N)$ we have

$$C_k(N'') = (jk\,.\,jk) \tag{17}$$

and

$$C_k(N') = C_k(N) - cC_k(N''), \tag{18}$$

where $C_k(N'')$ is the complexity of N'' at the vertex obtained by identifying P_j and P_k. It is clear that with an analogous interpretation of $U_k(N'')$ we have also

$$U_k(N') = U_k(N) - cU_k(N''). \tag{19}$$

3·6. THEOREM.

$$C_k(N) = U_k(N). \tag{20}$$

If N has just one vertex P_k, $C_k(N) = 1$ (§ 3·1), but $U_k(N)$ is undefined. We *define* $U_k(N)$ to be 1 so that the theorem may be true in this case.

If N has just two vertices P_j and P_k, we have $C_k(N) = U_k(N) = -c_{jk}$. It is now only necessary to consider the case in which N has at least three vertices.

If P_k is not the negative end of any edge we have at once $U_k(N) = 0$. Moreover, with the possible exception of c_{kk} the kth column of $\{c_{rs}\}$ consists entirely of 0's, so that $C_k(N)$, which can be defined as the cofactor of the kth element of another column, is 0. So the theorem is true in this case.

If P_k is the negative end of an edge L, we define N' and N'' as in § 3·5. By (18) and (19) the theorem will be true for N at P_k if it is true for N' and N'' at P_k. As N' and N'' each have fewer edges than N, the general result follows by induction over the number of edges of N.

3·7. We say that the network N is *simple* if the conductance of each of its edges is 1 and if also each vertex has just as many edges directed to it as directed from it. Thus any simple N satisfies (8) and so has the same complexity $C = C(N)$ at each vertex (by § 3·1).

3·8. THEOREM. *If N is simple and connected, and has at least two vertices, and if P_k is any one of its vertices, then N has a subtree which converges to P_k.*

LEMMA. *If P_r, P_s are any two distinct vertices of N, then P_r and P_s can be joined in N by a simple arc Λ such that each vertex of Λ other than P_s is the positive end of just one edge in Λ.*

We call such an arc Λ a *directed arc from P_r to P_s*.

If the lemma is true for a particular pair P_r, P_s, we say that P_s is *accessible* from P_r.

Let X be the set of all vertices of N accessible from P_r, together with P_r itself, and let Y be the set of all other vertices of N. If Y is not null, then since N is connected there must be at least one edge having one end in X and the other in Y. Clearly the end in X must be the negative end for each such edge (by the definitions of X and Y). It follows that the number of edges whose positive end is in Y exceeds the number whose negative end is in Y. Hence some vertex of Y is the positive end of more edges than have it as negative end, contrary to the definition of a simple network. Hence Y must be null and so the lemma is true.

Let the vertices of N other than P_k be enumerated as $Q_1, Q_2, \ldots, Q_{p-1}$. Let $\Lambda_1, \Lambda_2, \ldots, \Lambda_{p-1}$ be directed arcs in N such that Λ_s is directed from Q_s to P_k. The existence of such arcs follows from the lemma.

We define $G_1, G_2, \ldots, G_{p-1}$ successively as follows: $G_1 = \Lambda_1$. G_{s+1} $(0 < s < p-1)$ is the union of G_s and that part of Λ_{s+1} which extends from Q_{s+1} to the first vertex of Λ_{s+1}, reckoning from Q_{s+1}, which is in G_s.

From this definition we find, by considering each G_s in turn, that G_s is a tree for each s. As G_{p-1} contains each vertex of N it is therefore a subtree of N. Also each vertex of G_s other than P_k is the positive end of just one edge of G_s, and P_k is not the positive end of any edge of G_s. Hence by § 3·4, G_{p-1} converges to P_k.

Corollary. *If N is simple and connected, then $C(N) > 0$.*

This follows from § 3·6.

3·9. From the above corollary, it follows that for simple networks we can replace (15) by

$$(qr\,.\,tu) + (rs\,.\,tu) = (qs\,.\,tu) \tag{21}$$

in closer analogy with the equations of (A).

If N is simple we call the oriented network obtained from it by reversing the orientation of each edge the *reversal* of N, and denote it by N^*. Clearly N^* is simple.

If we distinguish quantities referring to N^* by an asterisk we have $c^*_{rs} = c_{sr}$, so that the matrix $\{c^*_{rs}\}$ is the transpose of $\{c_{rs}\}$. From this it follows that

$$C(N^*) = C(N) \tag{22}$$

and

$$(rs\,.\,tu)^* = (tu\,.\,rs). \tag{23}$$

The reversal has no analogy in the theory of (A).

3·10. Consider the 2-complex $N^\sigma(T)$ of § 2·9. We define the conductance of each of its edges to be 1. Then the 1-section of $N^\sigma(T)$ (i.e. the network defined by its edges and vertices) is simple, by § 2·9. The quantity c_{rs} for this network is clearly the quantity denoted in § 2 by c^σ_{rs}.

Suppose that S_σ is p^σ_u and that Σ_σ is p^σ_t. Then by applying § 3·3 to equations (4) we deduce that

$$V^\sigma_r - V^\sigma_s = \frac{X}{C^\sigma}(tu\,.\,rs), \tag{24}$$

where C^σ is the complexity of the 1-section of $N^\sigma(T)$.

It is convenient to measure the size of Δ in such units that $X = C^\sigma$. We then have

$$V^\sigma_r - V^\sigma_s = (tu\,.\,rs). \tag{25}$$

We call the corresponding values of the x_r the *full sizes* of the elements of T *with* *respect to* σ. It will, however, be shown later that $C^1 = C^2 = C^3$, so that the full sizes are in fact independent of σ. The full sizes of the elements of Δ are, by (25), integers. We thus have the

THEOREM. *The length of the side of* Δ, *and the sizes of the elements of* T *are commensurable.*

4. CONSTRUCTION OF A TRIANGULATION FROM A BICUBICAL MAP

4·1. A bicubical map M may be defined as follows. M is a finite cellular 2-complex which is a dissection of Z^2, and which satisfies the following conditions:

(i) Each vertex is incident with just three 2-cells and therefore with just three edges, and

(ii) The vertices of M fall into two mutually exclusive classes **E** and **F** such that each edge is incident with just one member of each class.

The bicubical map will be called *admissible* if it also satisfies the condition:

(iii) Each (closed) edge meets the boundaries of just four 2-cells.

Since the 2-cells are simple polygons it follows that the 1-section of M is connected.

The map $M(T)$ of § 2 is an admissible bicubical map, by § 2·7.

4·2. A 3-*colouring* of a bicubical map M is a partitioning of its 2-cells among three mutually exclusive classes, called *colour-classes*, so that no two members of the same colour-class have a side in common.

THEOREM. *A bicubical map* M *has just one* 3-*colouring*.

Let the edges of M be oriented so that the positive end of each is in **E**, and the 2-cells so that the 2-chain in which each coefficient is unity is a 2-cycle†. Then the 1-chain on M in which the coefficient of each edge is the residue 1 mod 3 is clearly a 1-cycle, K^1 say. Since M is a 2-sphere this 1-cycle bounds a 2-chain K^2 over the additive group of residues modulo 3 on M. We classify the 2-cells of M according to their coefficients in K^2. We thus obtain a 3-colouring of M, for if the two 2-cells incident with any edge have the same coefficient in K^2, that edge must have coefficient 0 in K^1.

It is easily seen that M has at most one 3-colouring. For when the three 2-cells incident with any particular vertex are assigned to their colour-classes, the assignments at each of the vertices joined to the first vertex by a single edge are determined.

4·3. Consider an admissible bicubical map M. We enumerate the members of **E** as $e_1, e_2, \ldots, e_n$ and the members of **F** as $f_0, f_1, \ldots, f_{n-1}$. That **F** has the same number of members as **E** follows as in § 2·3. We denote the three colour-classes of the bicubical map by $K(1)$, $K(2)$, $K(3)$ and enumerate the members of $K(\sigma)$ as P_i^σ $(i = 1, 2, \ldots, m_i)$. It will be seen that this notation agrees with that of § 2 for the bicubical maps considered there. We define $L(\sigma)$ as in § 2·6. We also define three 2-complexes $N^\sigma(M)$ just as we defined the $N^\sigma(T)$ in § 2·9. If M is the bicubical map of a triangulation T we can clearly suppose the notation adjusted so that $N^\sigma(M) = N^\sigma(T)$.

† For definitions of the terms of combinatorial topology used here, reference may be made to Seifert and Threlfall, *Lehrbuch der Topologie* (Leipzig and Berlin, 1934), to Alexandroff and Hopf, *Topologie* (Berlin, 1935), or to Lefschetz, *Algebraic Topology*, American Math. Soc. Colloquium publications, vol. 27. Here we use the results of Chapter V, § 3 of the second of these works.

We suppose hereafter that the enumeration of the P_i^σ is such that the member of $K(\sigma)$ incident with f_0 is P_1^σ for each σ.

We define λ_{rs}^σ to be 1 if P_r^σ is incident with e_s and 0 otherwise.

4·4. Consider the equations

$$\sum_s \lambda_{rs}^\sigma y_s = 0 \quad (\text{all } r \neq 1, \text{ all } \sigma). \tag{26}$$

M has $2n$ vertices, $3n$ edges and $(n+2)$ 2-cells (as in § 2·12). Hence the equations (26) are $n-1$ in number and involve just n unknowns y_s. Since the equations are linear and homogeneous it follows that they have a solution in which the y_s are real and not all zero. Henceforth by 'the y_s' we shall mean a particular solution of this kind. If $M = M(T)$ for some triangulation T we get such a solution by putting $y_s = x_s$ (f_0 representing the exterior triplet) by § 2·11.

4·5. We denote by L_r^σ the edge of $L(\sigma)$ which is incident with e_r, and by f_r^σ the member of **F** which is incident with L_r^σ. Let

$$\Gamma^1 = \sum_{\sigma, r} g_r^\sigma L_r^\sigma \tag{27}$$

be any 1-cycle on M with rational integer coefficients g_r^σ such that $g_r^\sigma = 0$ when L_r^σ is incident with f_0. Orientation is defined as in § 4·2.

Then Γ^1 bounds a 2-chain on M in which the coefficients of the P_1^σ are all equal, and therefore (by adding a 2-cycle) a 2-chain

$$H^2 = \sum_{\sigma, r} h_r^\sigma P_r^\sigma, \tag{28}$$

in which the h_r^σ are rational integers such that $h_r^\sigma = 0$ whenever $r = 1$.

THEOREM. *For each* 1-*cycle* Γ^1 *of the form* (27)

$$\sum_{\sigma, r} g_r^\sigma \omega^\sigma y_r = 0, \tag{29}$$

where ω *is an imaginary cube root of unity.*

By the foregoing considerations it will suffice to prove this for the case in which Γ^1 bounds a 2-cell P_q^ρ not incident with f_0. For by (28) every Γ^1 of the form (27) is a linear combination of 1-cycles of this type.

Suppose then that Γ^1 bounds P_q^ρ and that the three colour-classes are $K(\rho)$, $K(\theta)$ and $K(\phi)$. Without loss of generality we can suppose that g_r^σ is 0 when L_r^σ is not incident with P_q^ρ, and equal to $+1$ or -1 when L_r^σ is incident with P_q^ρ according as σ is θ or ϕ. Then for Γ^1 we have, by (26),

$$\sum_{\sigma, r} g_r^\sigma \omega^\sigma y_r = \sum_r \lambda_{qr}^\rho (\omega^\theta - \omega^\phi) y_r = (\omega^\theta - \omega^\phi) \sum_r \lambda_{qr}^\rho y_r = 0.$$

We have used the evident fact that edges of $L(\theta)$ must alternate with edges of $L(\phi)$ in the boundary of P_q^ρ.

The theorem follows.

4·6. Let A be any vertex of M other than f_0. Then if B is any other vertex of M not f_0 we can, since M is connected, find a 1-chain Y whose combinatorial boundary is the 0-chain $B-A$. (We adopt the convention that the combinatorial boundary of L_r^σ is $f_r^\sigma - e_r$.) We may suppose that the edges of M incident with f_0 have zero coefficients in Y. We can arrange this if necessary by adding a suitable linear combination of the boundaries of the P_1^σ.

Suppose that $$Y = \sum_{\sigma, r} v_r^\sigma L_r^\sigma.$$

Then we define the *potential* $\pi(B)$ of B by
$$\pi(B) = \sum_{\sigma, r} \omega^\sigma v_r^\sigma y_r. \tag{30}$$

By equation (29) it follows that the same value of $\pi(B)$ is obtained for each possible Y. We note that $\pi(A) = 0$. Evidently differences of potential are independent of the choice of A.

Considering the edge L_r^σ we find that
$$\pi(f_r^\sigma) - \pi(e_r) = \omega^\sigma y_r, \tag{31}$$
provided that f_r^σ is not f_0. If we try to calculate $\pi(f_0)$ from the edge of $L(\sigma)$ which is incident with f_0 by (31), we shall obtain a result $\pi^\sigma(f_0)$, but this will not necessarily have the same value for each σ. For convenience we also write, for each $r > 0$ and for each σ, $\pi(f_r^\sigma) = \pi^\sigma(f_r^\sigma)$.

4·7. Let the complex numbers $\pi(e_r)$, $\pi^\sigma(f_r^\sigma)$ be represented by points in the Argand plane. The four points $\pi(e_r)$, $\pi^1(f_r^1)$, $\pi^2(f_r^2)$ and $\pi^3(f_r^3)$ coincide when $y_r = 0$, but otherwise the first is the centre of an equilateral triangle of which the other three are the vertices (by (31)). We denote this (closed) triangle by E_r and call it an *element*.

The side of E_r opposite $\pi^\sigma(f_r^\sigma)$ is the set of points
$$(\tfrac{1}{2} + \alpha)\, \pi^{\sigma+1}(f_r^{\sigma+1}) + (\tfrac{1}{2} - \alpha)\, \pi^{\sigma+2}(f_r^{\sigma+2}), \tag{32}$$
where α takes all real values in the range $-\frac{1}{2} \leqslant \alpha \leqslant \frac{1}{2}$. The index $\sigma + 1$ or $\sigma + 2$, if greater than 3, is taken as equivalent to $\sigma - 2$ or $\sigma - 1$ respectively. Now (32) is the set
$$\pi(e_r) - \tfrac{1}{2}\omega^\sigma y_r + (\omega^{\sigma+1} - \omega^{\sigma+2})\, y_r \alpha, \tag{33}$$
by (31). This is a segment of a straight line $Y + (\omega^{\sigma+1} - \omega^{\sigma+2})\beta$, where Y is a constant and β is a real parameter. Let the value of β at any point ξ in this line be denoted by $\beta(\xi)$. Then evidently
$$\beta(\pi^{\sigma+1}(f_r^{\sigma+1})) - \beta(\pi^{\sigma+2}(f_r^{\sigma+2})) = y_r. \tag{34}$$

Consider a particular 2-cell P_k^σ of M. For each e_r incident with P_k^σ and such that $y_r \neq 0$, there is an element E_r. The sides of these E_r opposite the vertices $\pi^\sigma(f_r^\sigma)$ are parallel segments (by (33)). Moreover, they are in the same straight line and form a connected set. For if f_0 is not incident with P_k^σ each of these E_r has a vertex in common with each of its neighbours in the cyclic sequence corresponding to the sequence of the e_r with $y_r \neq 0$ in the boundary of P_k^σ. If f_0 is incident with P_k^σ the same rule holds except that the two E_r corresponding to the two e_r on either side of f_0 need not have a vertex corresponding to f_0 in common. In either case the union of the sides of the E_r opposite the vertices $\pi^\sigma(f_r^\sigma)$ is connected, and so is a single straight segment; we denote it by p_k^σ.

The segments $(0, \omega^\sigma)$ and $(0, (\omega^{\sigma+1} - \omega^{\sigma+2}))$ are perpendicular. Hence by (33) if $y_r \neq 0$ the point $\pi(e_r)$ and therefore the element E_r lies on one side or the other of p_k^σ according as y_r is positive or negative.

4·8. Let ξ be any point in the Argand plane not contained in any of the segments p_k^σ. Then we call the number of elements of which it is an interior point the *degree* of ξ. We denote this by $\delta(\xi)$.

The segments p_k^σ are finite in number; the complement of their union must therefore have only a finite number of components. Let these be enumerated as $B_1, B_2, \ldots, B_u$, say. All the points of any particular one of these evidently have the same degree.

Consider a particular segment p_i^ρ and let ζ be a point in p_i^ρ which is not the intersection of any two of the p_k^σ. All but a finite number of the points of p_i^ρ must be of this form.

By (33) p_i^ρ is a segment of a straight line $Y+(\omega^{\rho+1}-\omega^{\rho+2})\beta$, where Y is a constant and β is a real parameter.

Consider the cyclic sequence of the vertices in the boundary of P_i^ρ. Since edges of $L(\rho+1)$ alternate with edges of $L(\rho+2)$ in the boundary of P_i^ρ, we can suppose that each of the e_r incident with P_i^ρ is immediately succeeded in the cyclic sequence by $f_r^{\rho+1}$ and immediately preceded by $f_r^{\rho+2}$.

Now ζ is in a side of E_r if and only if either

$$\text{(i)} \quad \beta(\pi^{\rho+1}(f_r^{\rho+1})) > \beta(\zeta) > \beta(\pi^{\rho+2}(f_r^{\rho+2}))$$

or

$$\text{(ii)} \quad \beta(\pi^{\rho+1}(f_r^{\rho+1})) < \beta(\zeta) < \beta(\pi^{\rho+2}(f_r^{\rho+2})).$$

E_r is on one side or the other of p_i^ρ according as y_r is positive or negative (by § 4·7), that is, according as (i) or (ii) holds (by (34)). But it is clear that the number of pairs $(f_r^{\rho+1}, f_r^{\rho+2})$ satisfying (i) is equal to the number satisfying (ii), save possibly in the case $i=1$, when P_i^ρ is incident with f_0. In that case if $\beta(\zeta)$ lies between $\beta(\pi^{\rho+1}(f_0))$ and $\beta(\pi^{\rho+2}(f_0))$ the numbers of pairs satisfying (i) and (ii) will differ by 1; otherwise they will be equal.

Now for each σ the segments p_1^σ, $p_1^{\sigma+1}$ have the point $\pi^{\sigma+2}(f_0)$ in common. For if e_s is the vertex of $\mathbf{E}$ which is joined to f_0 by an edge of $L(\sigma+2)$, E_s has a side in each of p_1^σ and $p_1^{\sigma+1}$ (definition of the p_k^σ), and the vertex of E_s opposite the third side is $\pi^{\sigma+2}(f_0)$. Hence either p_1^1, p_1^2 and p_1^3 contain the sides of an equilateral triangle Δ whose vertices are the points $\pi^\sigma(f_0)$, or else they have a common point. In the latter case we say that Δ coincides with this point and has side zero.

There must be some point η in a B_s outside Δ such that $\delta(\eta) = 0$, since the outside of Δ is infinite in area. It follows by the above considerations that $\delta(\eta) = 0$ when η is in a B_s outside Δ, and $\delta(\eta) = 1$ when η is in a B_s inside Δ. Consequently the E_r are the elements of a triangulation $T = T(M, f_0)$ of the equilateral triangle Δ. (Since not all the y_r are zero, Δ cannot in fact have side zero.)

4·9. The triangulation T is not essentially altered if we multiply each y_r by -1. For the effect of this on the potentials $\pi(B)$ is merely to multiply them by -1 (by (30)). The new y_r still satisfy (26).

THEOREM. *We can arrange, by multiplying all the* y_r *by* -1, *if necessary, that whenever* $y_r \neq 0$, $y_r = x_r/\sqrt{3}$, x_r *being the size of the element* E_r.

First, for any E_r we find from (31) that

$$x_r = \pm\,|\,\pi(f_r^\sigma)-\pi(f_r^{\sigma+1})\,| = \pm\,y_r\,|\,\omega^\sigma-\omega^{\sigma+1}\,| = \pm\sqrt{3}y_r.$$

For any particular element E_s we arrange, by changing the sign of all the y_r if necessary, that x_s and y_s have the same sign.

But it is clear from (31) that all the positive elements have the same sign for y_r and all the negative elements have the opposite sign for y_r. Hence x_r and y_r now have the same sign for each E_r.

We assume henceforth that y_r is made to satisfy this condition.

4·10. We define a matrix c_{rs}^{σ} for $N^{\sigma}(M)$ just as we defined it for $N^{\sigma}(T)$ in § 2·10. Equations (1) and (2) then hold for $N^{\sigma}(M)$.

We define V_r^{σ} to be $2/\sqrt{3}$ times the distance of p_r^{σ} from p_1^{σ} measured positively towards $\pi^{\sigma}(f_0)$.

It is easily verified, by an argument similar to that of § 2·11, that equations (4) and (5) are also true for $N^{\sigma}(M)$. Applying the theory of § 3 (as in § 3·10) we then find that the new quantities V_r^{σ} satisfy (24), C^{σ} and $(tu\,.\,rs)$ being defined in terms of $N^{\sigma}(M)$. It follows that when X is given, the differences $V_r^{\sigma} - V_s^{\sigma}$ and therefore the quantities x_r and so also the y_r are fixed uniquely. Thus the solution of (26) for the y_r is unique apart from multiplication by an arbitrary constant.

When we measure the side of Δ in such units that $X = C^{\sigma}$, (24) reduces to (25), and so the sizes of the elements become integers. We have still to prove that C^{σ} is the same for each σ.

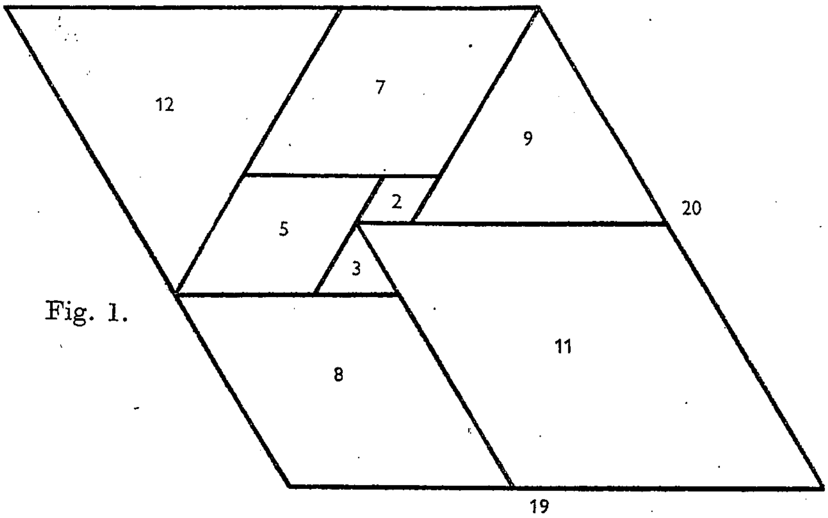

Fig. 1.

4·11. From a given bicubical map M we can in general derive several distinct triangulations by taking different members of **F** as f_0. Further, we can interchange the members of **E** and **F** and then derive another set of triangulations. This operation evidently replaces $N^{\sigma}(M)$ by its reversal.

It is therefore possible to determine all the triangulations of equilateral triangles of a reasonably low order n by first listing the bicubical maps of $2n$ vertices and then deriving from each the corresponding triangulations. Alternatively, we may prefer to list the maps $N^{\sigma}(M)$, characterized by the property that at each vertex edges directed to that vertex alternate with edges directed from it. (It is easily verified that the structure of $N^{\sigma}(M)$ completely determines that of M.) This is in closer analogy with the methods of (A). Not all the triangulations so obtained will be of the nth order necessarily, since in particular cases some of the y_r may vanish, but any triangulation T of the nth order will clearly be obtained from the corresponding map $M(T)$. I find that the two simplest perfect triangulations of equilateral triangles are those which can be obtained from the parallelogram of Fig. 1 by erecting equilateral triangles on two of its sides meeting at an acute angle†.

† In Figs. 1–3, any pair of elements having a common side is represented as a rhombus. The numbers represent the lengths of the sides of the containing polygons, or of the dissected figure.

5. Triangulations of parallelograms

5·1. The theory of triangulations of equilateral triangles is easily modified to cover that of triangulations of parallelograms (with angles of $\frac{1}{3}\pi$ and $\frac{2}{3}\pi$). I find that the simplest perfect parallelogram of order $n > 2$ is that of Fig. 1.

Given any triangulation T of order n of a parallelogram Π we can erect two new elements E_{n+1}, E_{n+2} on two of its sides adjacent to an acute angle and so obtain a triangulation T' of order $n+2$ of an equilateral triangle Δ.

For Δ we will define S_1 to be the side meeting both E_{n+1} and E_{n+2}. Then in $M(T')$ the 2-cell Q corresponding to S_1 is a quadrilateral, for it is incident with e_{n+1} and e_{n+2} but with no other member of **E**. One of the members of **F** incident with Q is the representative vertex f_0 of the exterior triplet.

Conversely, with the notation of § 4, suppose that P_1^1 is a 2-cell of M which is a quadrilateral, and which is incident with f_0. Let the three members of **E** joined to f_0 by 1-simplexes be e_r, e_{r+1} and e_{r+2}, the two latter being incident with P_1^1. Let the member of $K(2)$ on the opposite side of the quadrilateral P_1^1 to P_1^2 be P_j^2. Then the triangulation $T(M, f_0)$, the side S_1 of Δ corresponding to P_1^1, contains sides of just two elements E_{r+1} and E_{r+2} of $T(M, f_0)$. Suppressing these two elements we obtain a triangulation of a parallelogram. The lengths of the sides of this parallelogram can be obtained in terms of $N^2(M)$ by using (24) with the interpretation of § 4·10. If we adopt the convention that the side of Δ is C^2 (the complexity of $N^2(M)$), they are $(1j\,.\,1j)$ and $C^2-(1j\,.\,1j)$.

5·2. We can obtain another triangulation of a parallelogram (Π^* say) by interchanging the members of the classes **E** and **F** and then taking the vertex denoted above by e_{r+1} to represent the exterior triplet of a corresponding triangulation of a triangle. For the quadrilateral P_1^1 represents a side of the triangulated triangle. In this operation on M, $N^2(M)$ is replaced by its reversal. As before, if the side of the triangle is $(C^2)^*$, the complexity of $(N^2(M))^*$, then the sides of Π^* are $(1j\,.\,1j)^*$ and $(C^2)^*-(1j\,.\,1j)^*$.

Thus the sides of Π^* have the same lengths as those of Π (by (22) and (23)). In general, therefore, given any triangulation of a parallelogram Π, we can find another triangulation of Π of the same or smaller order. (Conceivably $(rs\,.\,tu)^*$ but not $(rs\,.\,tu)$ may vanish). Figs. 2 and 3 show two perfect triangulations related in this way.

5·3. Evidently to any 2-cell of M which is a quadrilateral not incident with f_0 there corresponds a pair of elements with a common side in the triangulation $T(M, f_0)$.

A case of particular interest arises when all the members of $K(1)$ are quadrilaterals. In any corresponding triangulated parallelogram the elements then fall into disjoint pairs, each pair constituting a rhombus, and the shorter diagonals of the rhombuses are all parallel. Such a dissection of a parallelogram into rhombuses is clearly equivalent to a dissection of a rectangle of the same side-lengths into squares. Conversely, by 'shearing' any squared rectangle we can obtain a dissection of a parallelogram into rhombuses, and we can relate this to an admissible bicubical map M in which all the members of $K(1)$ are quadrilaterals.

It is easily verified that in this case $N^2(M)$ and $N^3(M)$ correspond to dual c-nets† associated with the squared rectangle. A 'wire' in such a c-net is represented by two oppositely directed edges with the same end-points and bounding a 2-sided 2-cell in the corresponding map $N^2(M)$ or $N^3(M)$. This 2-cell is a quadrilateral of $K(1)$ in M.

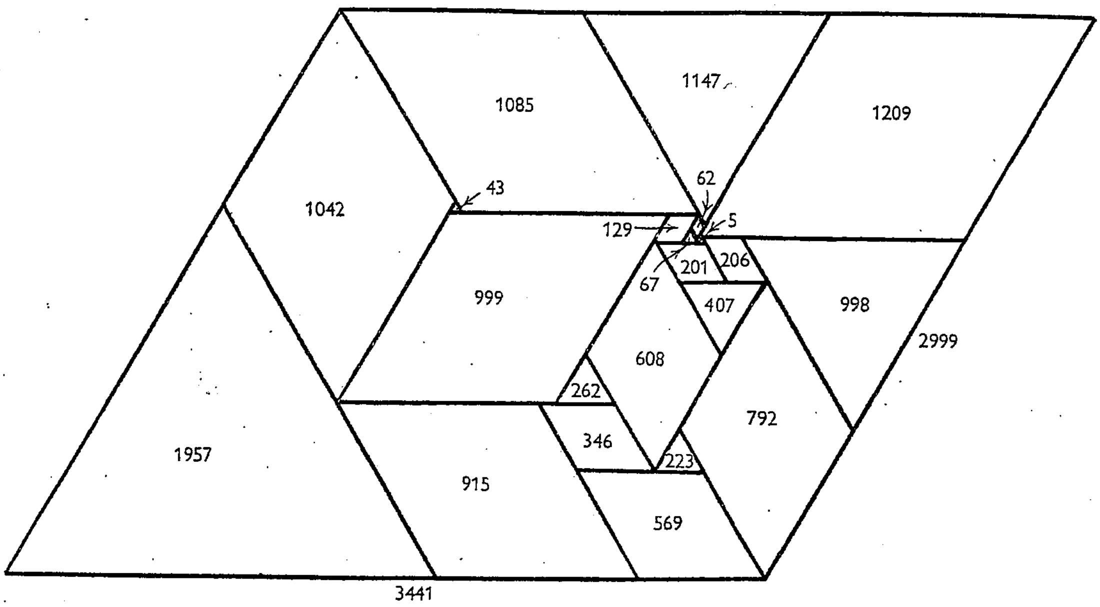

Fig. 2.

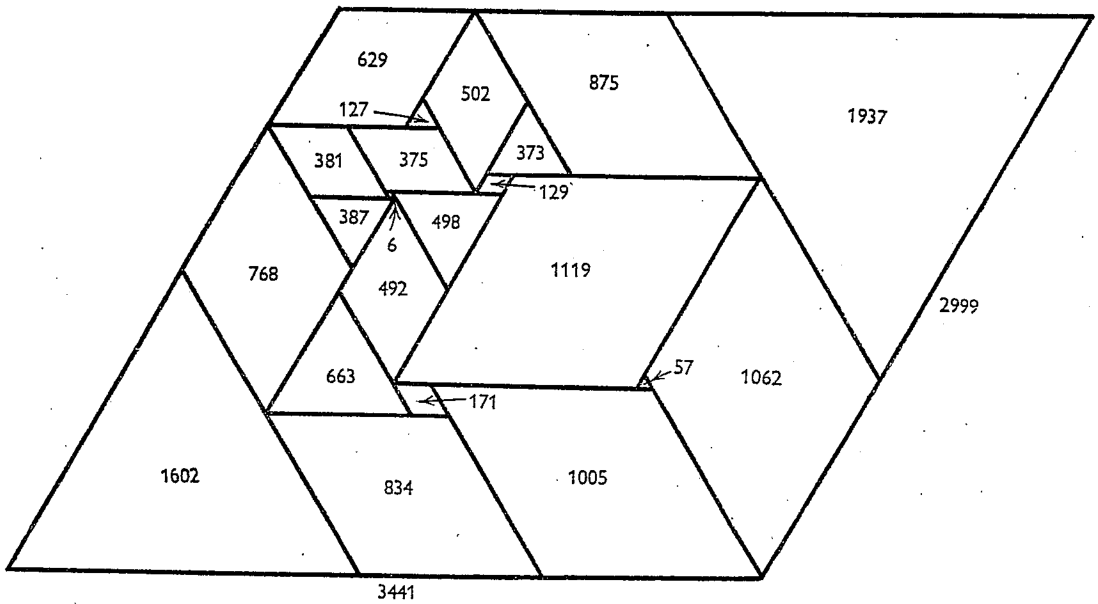

Fig. 3.

The quantities $(rs\,.\,tu)$ for $N^2(M)$ and $N^3(M)$ become identical with the transpedances of the corresponding c-nets.

This is why we describe the relationship between the three maps $N^\sigma(M)$ as a generalization of topological duality on the sphere. Further justification is given by the theorem of § 6.

† (A) § (3·3).

Dr C. A. B. Smith points out that the results of (A) enable us to prove the following

THEOREM. *A regular hexagon can be dissected into rhombuses, all of different sizes.*

We first dissect the hexagon into three equal rhombuses by joining the centre to alternate vertices. We obtain dissections of two of these by 'shearing' perfect squares. If the two perfect squares are 'totally different' ((A), § (8·1)) the hexagon will now be dissected into unequal rhombuses (see (A), § (9·11)).

COROLLARY. *There exists a perfect triangulation of a regular hexagon.*

We dissect the hexagon into unequal rhombuses and then dissect each rhombus into two equilateral triangles. One of these triangles will be a positive, the other a negative element of the triangulation.

6. THE COMPLEXITIES C^σ

6·1. In § 4 we showed that in the triangulation $T(M, f_0)$ obtained from a bicubical map M the sizes of the elements all become integers when the side of the complete triangle is made equal to C^σ. This suggests that

$$C^1 = C^2 = C^3. \tag{35}$$

The object of the present section is to prove (35) for every bicubical map M. Equation (35) is the analogue of Theorem (3·25) of (A). This applies to dual undirected networks, whereas (35) applies to trial directed ones.

6·2. In any bicubical map M members of the three classes $K(1)$, $K(2)$ and $K(3)$ occur in the same cyclic order at each member of $\mathbf{E}$, and in the opposite cyclic order at each member of $\mathbf{F}$ (with respect to a fixed positive sense of rotation in Z^2). To prove this we have only to observe that the orders at the two ends of any given edge must be opposite.

6·3. Select any particular member e_k of $\mathbf{E}$.

Suppose that there are given sets D^1, D^2, D^3 of edges of $N^1(M)$, $N^2(M)$ and $N^3(M)$ respectively, together forming a set D such that (i) if P is any 2-cell of M not incident with e_k, then just one member of D has P as its positive end in the appropriate $N^\sigma(M)$, (ii) if P is any 2-cell of M incident with e_k, then P is not the positive end of any member of D in an $N^\sigma(M)$, (iii) each member of $\mathbf{E}$ other than e_k is the positive end in M of just one member of D, and (iv) e_k is not incident in M with any member of D.

We denote by R^σ the subnetwork (see § 3·4) of the 1-section of $N^\sigma(M)$ whose edges are the members of D^σ. (We call it also a subnetwork of $N^\sigma(M)$.)

THEOREM. *For each D satisfying the above conditions, and for each σ, R^σ is a subtree of $N^\sigma(M)$ converging to that vertex of $N^\sigma(M)$ which, as a 2-cell of M, is incident with e_k.*

We suppose this vertex denoted by P_1^σ (contrary to our former convention).

Assume that for some σ, R^σ contains a simple closed curve Γ.

Then each vertex of $N^\sigma(M)$ in Γ is the positive end of just one edge of Γ. Otherwise some vertex will be the positive end of two edges of Γ, contrary to (i) and (ii). Hence by (ii) e_k is not incident in M with any vertex of Γ.

Clearly by making joins inside some 2-cells of M (vertices of Γ) we can obtain a simple closed curve Γ' in M which contains every edge of M which is an edge of Γ and otherwise lies entirely in the interiors of the 2-cells of $K(\sigma)$. By the preceding paragraph,

Γ' does not contain e_k, and members of **E** and **F** occur alternately in Γ' (i.e. the edges of M in Γ' are all directed the same way round Γ').

From this, with the help of (6·2) we deduce that the 2-cells of M contained in a particular residual domain X of Γ' and meeting Γ' all belong to the same colour-class $K(\rho)$ say, which is not $K(\sigma)$. We will take X to be that residual domain of Γ' which does not contain e_k.

Now any edge of D^ρ having its positive end (as an edge of $N^\rho(M)$) contained in X also has its negative end (as an edge of $N^\rho(M)$) contained in X and is itself contained in X. Otherwise it would have to be incident with a vertex of M contained in Γ'. This vertex would have to belong to **F** (by (i) and (ii)), which would imply that some 2-cell† contained in X and not of colour σ or ρ would meet Γ'.

Each edge of M in Γ' is incident with a 2-cell contained in X. Hence X contains at least one 2-cell of $K(\rho)$.

Let H be the part of $N^\rho(M)$ consisting of those edges and vertices which, as edges and 2-cells of M, are contained in X. Let H_0 be obtained from H by suppressing all the edges except those whose positive ends are in X and which belong to D^ρ. By the preceding considerations H_0 is a network which with any vertex of $N^\rho(M)$ contains also the edge, if any, of D^ρ which has that vertex as positive end. Since e_k is not in X, it follows from (i) that H_0 contains precisely as many edges as vertices. So, using the formula‡

$$p_1(L) - p_0(L) = \alpha_1(L) - \alpha_0(L), \tag{36}$$

L being any network, $p_i(L)$ the Betti number of dimension i of L, and $\alpha_1(L)$ and $\alpha_0(L)$ the numbers of edges and vertices of L respectively, we see that $p_1(H_0) = p_0(H_0) > 0$, so that H_0 must contain a simple closed curve.

We thus deduce that, given any simple closed curve Γ' in M containing one or more edges of D^σ, for some particular σ, and otherwise lying entirely in the interiors of the 2-cells of $K(\sigma)$, we can find another such curve, corresponding to a different value of σ, which is separated from e_k by Γ'. But this implies the existence of an infinity of such curves, of which no two intersect. This contradicts our requirement that the edges of M are finite in number.

It follows that R^σ has no simple closed curve. It has just one more vertex than edge (by (iii) and (iv)). Hence by (36), $p_0(R^\sigma) = 1$, and so R^σ is connected. Hence R^σ is a subtree of $N^\sigma(M)$. Another application of (i) and (ii) shows that R^σ converges to P_1^σ.

6·4. THEOREM. *Let R^σ be any subtree of $N^\sigma(M)$ converging to P_1^σ. Then there is just one set D satisfying* (i)–(iv) *such that the edges belonging to D^σ are the edges of R^σ.*

We define a sequence $Y(0)$, $Y(1)$, $Y(2)$, ... of sets of edges of M in the following way.

We first agree that any 2-cell P of M not incident with any member of **E** which is the positive end of a member of $Y(i)$ of the same colour as P shall be called *i-unused*. Further we agree that any member of **E** which is incident with no member of $Y(i)$ shall be called *i-unused*.

Given $Y(i)$ we choose, if possible, an i-unused 2-cell P, not incident with e_k, which is

† The 2-cell is that one incident with the given edge of D^ρ and not of colour σ.

‡ Seifert and Threlfall, p. 87 (Example 3).

incident with just one i-unused member e_{r_i} of $\mathbf{E}$. We adjoin to $Y(i)$ the edge incident with e_{r_i} and having the same colour as P. We take the resulting set as $Y(i+1)$. We define $Y(0)$ to be the set of edges of R^σ and continue the above process until it terminates, with $Y(m)$ say. It follows from (i) that any set D which contains $Y(0)$ contains also $Y(m)$.

All the 0-unused, and therefore all the m-unused 2-cells of M not incident with e_k have colours other than σ, since R^σ is a subtree of $N^\sigma(M)$.

We can show that each m-unused member of $\mathbf{E}$ other than e_k is incident with just two m-unused 2-cells. For let e_j be any m-unused member of $\mathbf{E}$ other than e_k (e_k is incident with three m-unused 2-cells). e_j is incident with just two 2-cells, P_1 and P_2 say, of colours other than σ. If either of these is incident with e_k it is m-unused by the construction. If one of them, P_1 say, is not incident with e_k and not m-unused, then for some $i < m$ it is i-unused but not $(i+1)$-unused. Hence $Y(i+1)$ was formed from $Y(i)$ by adding an edge incident with an i-unused vertex e_s incident with P_1 and not identical with e_j. Since e_j is also i-unused and incident with P_1 this was contrary to the conditions of the construction.

Construct a network in which the vertices are the m-unused members of $\mathbf{E}$, together with a set consisting of one interior point from each m-unused 2-cell other than P_1^σ, and in which the edges are simple arcs contained in the m-unused 2-cells. In each m-unused 2-cell there is just one such arc for each incident m-unused member of $\mathbf{E}$ joining this member of $\mathbf{E}$ to the chosen interior point. We can suppose that no two of these arcs in any one 2-cell intersect. Denote this network by G. Let G' be obtained from G by suppressing all the isolated vertices. Let the number of m-unused vertices of M be p, and the number of other vertices of G' be q. Then the number of edges of G' is $2p$. Also it is not less than $2(q-2)+2 = 2(q-1)$ with equality possible only when no 2-cell incident with e_k and not of colour σ is incident with any other m-unused member of $\mathbf{E}$ and when also no m-unused 2-cell is incident with more than two m-unused members of $\mathbf{E}$. (The P_1^σ are the only 2-cells which can be at once m-unused and incident with just one m-unused member of $\mathbf{E}$; otherwise we could construct a $Y(m+1)$.) Hence

$$p \geq q-1 \tag{37}$$

with equality only under the above conditions.

Now G' contains no simple closed curve. For if it contains a simple closed curve Γ, each component of the complement of Γ in Z^2 contains a member of F, and therefore a 2-cell of colour σ. The union of the (open) 2-cells of $K(\sigma)$ and the closures in M of the edges which are edges of R^σ is a connected set (R^σ is connected). But it does not meet Γ. This is absurd since we have shown that it meets each component of the complement of Γ. Hence if the number of components of G' is $p_0(G')$, we can by applying (36) to G' deduce that

$$2p = (p+q) - p_0(G').$$

Hence $p_0(G') = q-p \leq 1$ (by (37)). As $p_0(G')$ must exceed 0, since G' certainly contains e_k, it follows that $p = q-1$. We have seen that this can be true only when e_k, together with its two incident edges of G' constitutes a complete component of G'. Since $p_0(G') = 1$ this component is the whole of G'. Hence we have the result

M contains no m-unused member of $\mathbf{E}$, *other than* e_k.

Now if M has n members of $\mathbf{E}$, the number of its 2-cells is $n+2$, by (4·4) and (2·12). It follows by the method of construction of $Y(m)$ that the number of m-unused 2-cells is $(n+2)-(n-1) = 3$. These three are of course those incident with e_k.

It follows that $Y(m)$ satisfies the conditions (i)–(iv). But we have shown that any set D satisfying (i)–(iv) and containing $Y(0)$ contains also $Y(m)$. The theorem follows.

6·5. By the results of §§ 6·3 and 6·4 we can arrange the subtrees of the maps $N^\sigma(M)$ converging to the vertices P_1^σ in disjoint sets of three so that just one member of each set is a subtree of $N^\rho(M)$ for each ρ. The three members of any one of these sets define a set D satisfying (i)–(iv). (35) follows immediately by § 3·6.

We have now obtained the following curious topological theorems.

(i) The number of subtrees of an $N^\sigma(M)$ converging to a fixed vertex is independent of the particular vertex chosen. (§ 3·7.)

(ii) The number of subtrees of an $N^\sigma(M)$ converging to a given vertex is equal to the number diverging from that vertex (by (22)).

(iii) The numbers of subtrees of $N^1(M)$, $N^2(M)$ and $N^3(M)$ converging to particular vertices P^1, P^2, P^3 respectively, are equal.

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