THE DISSECTION OF EQUILATERAL TRIANGLES INTO EQUILATERAL TRIANGLES

By W. T. TUTTE

Received 10 December 1947

1. Introduction

In a previous joint paper (‘The dissection of rectangles into squares’, by R. L. Brooks, C. A. B. Smith, A. H. Stone and W. T. Tutte, Duke Math. J. 7 (1940), 312–40), hereafter referred to as (A) for brevity, it was shown that it is possible to dissect a square into smaller unequal squares in an infinite number of ways. The basis of the theory was the association with any rectangle or square dissected into squares of an electrical network obeying Kirchhoff’s laws. The present paper is concerned with the similar problem of dissecting a figure into equilateral triangles. We make use of an analogue of the electrical network in which the ‘currents’ obey laws similar to but not identical with those of Kirchhoff. As a generalization of topological duality in the sphere we find that these networks occur in triplets of ‘trial networks’ $N^1$, $N^2$, $N^3$. We find that it is impossible to dissect a triangle into unequal equilateral triangles but that a dissection is possible into triangles and rhombuses so that no two of these figures have equal sides. Most of the theorems of paper (A) are special cases of those proved below.

We define a triangulation of order $n$ of any region and in particular of an equilateral triangle $\Delta$ as a dissection of the region into $n > 1$ closed equilateral triangles, $E_1, E_2, \ldots, E_n$, called the elements of the triangulation, which between them completely fill the region and which do not overlap except at their boundaries. It is evident that in any such dissection of $\Delta$ the elements fall into two mutually exclusive classes, those placed similarly to $\Delta$ which will be called positive elements, and those placed similarly to the triangle formed by rotating $\Delta$ through an angle $\pi$ which will be called negative elements. The size $x_r$ of the element $E_r$ is defined to be the length of the side of $E_r$ taken with a positive or negative sign according as $E_r$ is a positive or a negative element. The triangulation of $\Delta$ is called perfect if no two elements have the same size. Thus a perfect triangulation has at most two elements with a given length of side, and if it has two then one is a positive and the other a negative element.

In §2 we describe a graphical representation $M(T)$ of a triangulation $T$ of $\Delta$. We call $M(T)$ the bicubical map of $T$. From $M(T)$ we get three networks $N^1(T)$, $N^2(T)$, $N^3(T)$. We show that with each of these three networks there is associated a set of equations, analogous to Kirchhoff’s laws, connecting the sizes of the elements of $T$. One result of this section is that in any triangulation of an equilateral triangle $\Delta$ there must be two elements with a common side, and therefore with equal and opposite sizes. In §3 we develop the theory of these equations and show that the sizes $x_r$ of the elements $E_r$ must be commensurable.

In §4 we give an independent definition of a bicubical map, and show that from any such bicubical map we can derive a triangulation of an equilateral triangle.
In § 5 we discuss triangulations of parallelograms, including as a special case squared rectangles \((A)\), Introduction. § 6 generalizes the main duality theorem (3-25) of \((A)\).

Most of the theorems proved below were discovered, or at least conjectured, during the researches which led to the results of paper \((A)\). Here they have been systematized and missing proofs, such as that of § 6, supplied. I am indebted to the other authors of \((A)\) for permission to use joint results and for helpful criticism during the preparation of the present paper. In particular I have to thank Dr C. A. B. Smith for the conception of the bicubical map of a triangulation, and the definitions of a ‘perfect triangulation’ and a ‘standardized matrix’.

2. Triangulations of triangles

2-1. Consider a triangulation \(T\) of order \(n\) of an equilateral triangle \(\Delta\) with elements \(E_1, E_2, \ldots, E_n\).

Let the sides of \(\Delta\) be \(S_1, S_2\) and \(S_3\). Clearly any side of an element of \(T\) is parallel to just one of these.

Let \(v\) be any vertex of an element \(E_r\) of \(\Delta\), but not a vertex of \(\Delta\). Since \(E_r\) occupies an angle \(\frac{1}{n}\pi\) at the vertex \(v\) we have two possibilities. Either six distinct elements have a vertex at \(v\) (when \(v\) will be called a cross), or just three distinct elements have a vertex at \(v\), the remaining angle of \(\pi\) at \(v\) being occupied by another element or by the exterior of \(\Delta\).

2-2. Theorem. For any triangulation \(T\) of an equilateral triangle \(\Delta\), we can find closed straight segments \(p^\sigma_i\) \((\sigma = 1, 2, 3; i = 1, 2, \ldots, m_\sigma)\), where \(m_1, m_2, m_3\) are some positive integers, such that (i) the union of the \(p^\sigma_i\) is the union of the sides of the \(E_r\), each side of each \(E_r\) being contained in some \(p^\sigma_i\), (ii) \(p^\sigma_i\) is parallel to the side \(S_\sigma\) of \(\Delta\), (iii) two distinct segments \(p^\sigma_i\) have at most one point in common, and (iv) if \(v\) is a vertex of \(E_r\) and not a vertex of \(\Delta\), then \(v\) is an interior point of just one of the segments \(p^\sigma_i\). (An interior point of \(p^\sigma_i\) is a point contained in, and not an end-point of, \(p^\sigma_i\).)

To prove this we consider the union \(U_\sigma\) of those sides of elements of \(T\) which are parallel to \(S_\sigma\). Its connected components are closed straight segments. Let us try taking these as the segments \(p^\sigma_i\) for each value of \(\sigma\). The segments \(p^\sigma_i\) thus defined evidently satisfy conditions (i) to (iv) save only that (iv) fails at each cross; a cross is an interior point of just three of these segments. Clearly, by subdividing at each cross any two of the segments intersecting there we obtain a set of segments satisfying the conditions of the theorem.

2-3. Suppose that we have a set of segments \(p^\sigma_i\) satisfying the conditions of § 2-2. Then if \(v\) is a vertex of an element of \(T\) but not a vertex of \(\Delta\), there is a unique segment \(p^\sigma_i\) having \(v\) as an interior point (§ 2-2 (iv)). We call this the bisector \(B(v)\) of \(v\). It defines two angles of \(\pi\) at \(v\), and each of these angles contains either 0 or 3 elements of \(T\) which have \(v\) as a vertex. We call a set of three elements occupying either of these angles a triplet of \(T\) with focus \(v\). The number of triplets of which \(v\) is a focus is thus two if \(v\) is a cross, and one otherwise. It is convenient to extend this definition by calling the set of three elements of \(T\) which meet vertices of \(\Delta\) a triplet (the exterior triplet of \(\Delta\)).
Dissection of equilateral triangles into equilateral triangles

Δ is regarded as imbedded in the closed plane $\mathbb{Z}^2$, and the point $J$ at infinity is called the focus of this special triplet.

We suppose the triplets enumerated as $F_0, F_1, \ldots, F_{q-1}$, the exterior triplet being $F_0$.

With these definitions it is clear that each element of $T$ belongs to just three triplets, one for each vertex.

We can represent the relations between the elements and triplets of $T$ by a linear graph $G(T)$. $G(T)$ has $n+q$ vertices, $e_1, e_2, \ldots, e_n$ corresponding to the elements, and $f_0, f_1, \ldots, f_{q-1}$ corresponding to the triplets. Two vertices are joined by at most one edge, and each edge has one end an $e_r$ and the other an $f_s$. There is an edge joining $e_r$ and $f_s$ if and only if $E_r$ belongs to the triplet $F_s$.

We see that each vertex of $G(T)$ is incident with just three edges. This property is expressed by saying that $G(T)$ is cubical. Moreover, the vertices of $G(T)$ fall into two exhaustive and mutually exclusive classes $E$ and $F$—the set of the $e_r$ and the set of the $f_s$ respectively—such that each edge is incident with one member of each class. We express both properties by saying that $G(T)$ is bicubical.

Since each edge joins two vertices, one of class $E$ and the other of class $F$, and each vertex is incident with just three edges, we see that $n$, the number of members of $E$ must equal $q$, the number of members of $F$.

$$n = q,$$

(1)

2-4. Theorem. $G(T)$ can be realized in the closed plane.

Take $e_r$ to be the centre of $E_r$. Make straight joins from $e_r$ to each of the vertices of $E_r$ (for each $r$). A straight join to a vertex of $\Delta$ is to be continued through that vertex to the point $J$ at infinity.

By this construction each element, and also the exterior of $\Delta$, is divided into three 3-sided regions. We call these regions subelements of $T$.

Consider the linear graph whose edges are these straight joins and whose vertices are the $e_r$ and the foci of the triplets. This would be a realization of $G(T)$ in $\mathbb{Z}^2$ were it not that each cross $X$ is the focus of two triplets. As the two triplets are separated at $X$ by the bisector $B(X)$ we can ‘pull apart’ the representative points of the two triplets at each cross and so obtain a realization of $G(T)$ in $\mathbb{Z}^2$. More precisely we choose some positive $\epsilon$ less than half the length of the side of the smallest element of $T$, and at each cross $X$ replace the part of the linear graph within $\epsilon$ of $X$ by the two arcs of the circle of radius $\epsilon$ and centre $X$ for which the radius makes an angle not less than $\frac{1}{3}\pi$ with the bisector of $X$. The midpoints of these two circular arcs are taken as the representative points $f_s$ corresponding to the two triplets.

2-5. Let $P_1^* \subseteq \mathbb{Z}^2$ be a closed polygon constructed from the union of those (closed) subelements of $T$ which have a side in $p_1^*$ by adding every point of $Z$ distinct not more than $\epsilon$ from any cross which is an interior point of $p_1^*$ and subtracting every point which is distant less than $\epsilon$ from any cross which is an end-point of $p_1^*$.

It is easily verified that $P_1^*$ is simply connected (its boundary being a simple closed curve), that no two of the $P_r^*$ have an interior point in common, and that the realization of $G(T)$ obtained in \S\ 2-4 contains the boundary, but no interior point of each $P_r^*$. 


the polygons $P_i^\sigma$ and whose 1-section (linear graph constituted by its edges and vertices) is our realization of $G(T)$. We call this 2-complex the *bicubical map* $M(T)$ of $T$.

2.6. The index $\sigma$ will be called the *colour* of $P_i^\sigma$. Since $E_\sigma$ has a side parallel to each $S_\sigma$, it follows from § 2.6 that $e_\sigma$ is incident with just one $P_i^\sigma$ of each colour $\sigma$. Consequently each edge of $M(T)$ is incident with two 2-cells of different colours. For each edge is incident with a member of $E$. The remaining colour will be called the *colour of the edge*. Since each edge is incident with a member of $F$ it follows further that the three 2-cells incident with any member of $F$ have three different colours. We denote by $K(\sigma)$ the class of the 2-cells of colour $\sigma$, and by $L(\sigma)$ the class of the edges of colour $\sigma$. By the above considerations the three classes $K(\sigma)$ are exhaustive and mutually exclusive, and so are the three classes $L(\sigma)$.

2.7. By the construction of § 2.5 it follows that $e_\sigma$ is incident with $P_i^\sigma$ if and only if $E_\sigma$ has a side contained in $p_i^\sigma$. Also $f_\sigma$ is incident with $P_i^\sigma$ if and only if the focus of $F_\sigma$ is in $p_i^\sigma$ and also $p_i^\sigma$ contains a side of some member of $F_\sigma$, save only that $f_\sigma$ is incident with each 2-cell of $M$ corresponding to a side $S_\sigma$ of $\Delta$. Henceforth we shall assume that $S_\sigma$ is $p_i^\sigma$.

We denote by $W_\sigma$ the vertex of $\Delta$ opposite $S_\sigma$.

**Theorem.** Each (closed) edge of $M(T)$ meets the boundaries of just four 2-cells.

Let $L$ be a 1-cell of $M(T)$ with end-points $e_\sigma$, $f_\sigma$, and colour $\sigma$. Let $P_i^\sigma$, $P_k^\sigma$ be the 2-cells of colour $\sigma$ incident with $e_\sigma$ and $f_\sigma$ respectively. Then they are distinct, for the element $E_\sigma$ (by the above considerations) has one side in $p_i^\sigma$ and the opposite vertex in $p_k^\sigma$ except when $p_k^\sigma$ is $S_\sigma$ and $E_\sigma$ has a vertex at $W_\sigma$.

So, besides the boundaries of its two incident 2-cells, $L$ meets the boundaries of just two other 2-cells.

2.8. We suppose henceforth that the edges of $M(T)$ are oriented, with positive ends $e_\sigma$ and negative ends $f_\sigma$. We say that an edge is *directed from* its positive end, and to its negative end.

2.9. For each $i$ ($1 \leq i \leq m_1$) let us identify all the points of the closed 2-cell $P_i^1$.

This process does not identify the end-points of any edge of $M(T)$ not incident with a member of $K(1)$. For by § 2.7 the positive end of such an edge would represent an element of $T$ having one side and also the opposite vertex in the same segment $p_i^1$, or else the edge would represent an element of $T$ with one side in $S_1$ and having the opposite vertex of $\Delta$ as a vertex, which is impossible since $n > 1$.

The result of the identifications is thus clearly a cellular 2-complex $N^1(T)$ which is a dissection of a space homeomorphic to $Z^2$. Its vertices are the $P_i^1$. Its 2-cells are the $P_i^2$ and the $P_k^2$. Its edges are the edges of $L(1)$. The edges and 2-cells have the same mutual incidence relations as they have in $M(T)$, and any one of them is incident with $P_i^1$ if and only if it is incident in $M(T)$ with an edge or vertex incident with $P_i^1$.

In a similar way, by operating on the 2-cells $P_j^2$ or $P_k^2$ instead of the $P_i^1$ we obtain 2-complexes $N^2(T)$ and $N^3(T)$ respectively. We say that the three $N^\sigma(T)$ constitute

† In the latter case $p_i^2$ and $p_i^2$ are distinct because, since $n > 1$, $E_\sigma$ is not the whole of $\Delta$. 
a set of trial 2-complexes provided that their edges are oriented according to the following rule: the positive (negative) end of an edge \( L \) of \( N^s(T) \), when regarded as a closed 2-cell of \( M(T) \), contains the positive (negative) end of the edge in \( M(T) \).

We shall see later that triality can be regarded as a generalization of topological duality in the 2-sphere.

We have seen in §2.7 that in \( M(T) \) the vertex \( e \) is incident with just one edge of each colour. We shall denote the edge of colour \( \sigma \) incident with \( e \) by \( L_\sigma \). Of the three edges incident with \( e \), only \( L_\sigma \) is an edge of \( N^\sigma(T) \). Thus to each element \( E_\sigma \) of \( T \) there corresponds a unique edge \( L_\sigma \) of \( N^\sigma(T) \).

The edges of \( N^\sigma(T) \) incident with the vertex \( P_\sigma \), taken in their cyclic order at \( P_\sigma \), are directed alternately to and from \( P_\sigma \). This follows from the fact that members of \( E \) and \( F \) must alternate in the boundary of the 2-cell \( P_\sigma \) of \( M(T) \), since \( G(T) \) is bicubical (§2.3).

2.10. We define a matrix \( \{c_{rs}\} \) as follows:

If \( r \neq s \), then \( c_{rs} \) is the number of edges of \( N^s(T) \) directed from \( P_r \) to \( P_s \), and \( c_{rs} \) is the number of edges of \( N^r(T) \) directed from \( P_r \) to \( P_s \). We note that

\[
\sum_{s} c^s_{rs} = 0 \quad \text{and} \quad \sum_{r} c^r_{rs} = 0. \tag{2}
\]

The first of these follows immediately from the definition of \( c^s_{rs} \). For the proof of the second we require also the result that the total number of edges directed to a given vertex of \( N^\sigma(T) \) is equal to the total number directed from that vertex (since edges of the two kinds alternate at the vertex).

2.11. Let \( W_\sigma \) be the vertex of \( \Delta \) opposite \( S_\sigma \). We can suppose that the element of \( T \) which meets \( W_\sigma \) is \( E_\sigma \). Let \( \Sigma_\sigma \) denote the side of \( E_\sigma \) opposite \( W_\sigma \).

It is evident that if \( p_\sigma \) is not \( S_\sigma \), the sum of the \( x_r \) (see Introduction) taken over all \( E_r \) having a side in \( p_\sigma \) is zero. But if \( p_\sigma \) is \( S_\sigma \) the sum is the length \( X \) of the side of \( \Delta \).

Putting this in terms of \( M(T) \) we find that the sum of the \( x_r \) taken over all \( e_r \) incident with a given \( p_\sigma \) is 0 or \( X \) according as \( p_\sigma \) is not or is incident with the special vertex \( f_\sigma \).

Let \( V_\sigma \) denote 2/3 times the distance of \( p_\sigma \) from \( S_\sigma \) measured positively towards \( W_\sigma \). Then if \( E_\sigma \) has a side in \( p_\sigma \) and the opposite vertex in \( p_\sigma \) we have

\[
x_{\sigma} = V_\sigma - V_{\sigma}. \tag{3}
\]

This equation applies for each \( E_\sigma \) except \( E_\sigma \) (\( W_\sigma \) is the only vertex of an element of \( T \) not in a \( p_\sigma \), for a fixed \( \sigma \)).

Using the above result for \( M(T) \), and (2) and (3), we find that

\[
\sum_{s} c^\sigma_{rs} (V_\sigma - V_{\sigma}) = \sum_{s} c^\sigma_{rs} V_\sigma = \begin{cases} 0 & \text{(if } p_\sigma \text{ is not } S_\sigma), \\ -X & \text{(if } p_\sigma \text{ is } S_\sigma), \end{cases} \tag{4}
\]

provided that \( p_\sigma \) is not \( \Sigma_\sigma \).

If \( p_\sigma \) is \( \Sigma_\sigma \) we find by analogous considerations that

\[
\sum_{s} c^\sigma_{rs} V_\sigma = x_\sigma - (0 - (X - x_\sigma)) = X. \tag{5}
\]

\( \dagger \) For \( r \neq s \), \( c_{rs} \) is thus minus the number of elements of \( T \) with bases on \( p_\sigma \) and vertices on \( p_\sigma \) (except that for the purposes of this enumeration the element having a vertex at \( W_\sigma \) is deemed to have it on \( S_\sigma \)).
Equations (4) and (5) constitute the set of linear equations associated with $N^e(T)$ which is referred to in the introduction. It will be shown in the next section that when $X$ is given they uniquely determine the differences of the $V^f \sigma$ (for any fixed $\sigma$) and so also the $x_\sigma$.

2-12. Theorem. In any triangulation $T$ of a triangle $\Delta$ some two elements have a side in common.

Let $\alpha_0$, $\alpha_1$ and $\alpha_2$ denote the number of vertices, edges and 2-cells of $M(T)$ respectively. By §2-3 we have $\alpha_0 = 2n$ and $\alpha_1 = 3n$. Hence by the Euler polyhedron formula it follows that $\alpha_2 = n + 2$.

Let $c_m$ be the number of 2-cells having $m$ sides. Since members of $E$ and members of $F$ alternate in the boundary of any 2-cell of $M(T)$, $c_m$ vanishes for all odd $m$. Hence

$$n + 2 = c_2 + c_4 + c_6 + \ldots = \alpha_2$$

and

$$3n = \frac{1}{2}(2c_2 + 4c_4 + 6c_6 + \ldots) = \alpha_1.$$ 

Hence

$$6 = 2c_2 + c_4 - c_6 - 2c_{10} - 3c_{12} - \ldots.$$ (6)

But by the theorem of §2-7, $c_2 = 0$, for a side of a 2-sided 2-cell could not satisfy that theorem. Hence by (6), $c_4 \geqslant 6$. Since $f_0$ is incident with just three 2-cells of $M(T)$, it follows that there is a 2-cell $P^f \sigma$ of $M(T)$ not incident with $f_0$ and having just four sides. Then $P^f \sigma$ is incident with just two of the $e_\sigma$. Since $P^f \sigma$ is not $S_\sigma$ ($P^f \sigma$ is not incident with $f_0$) it follows that $p^f \sigma$ is a side of each of the two corresponding elements $E_\sigma$.

3. The metrical properties of triangulations

3-1. Let $N$ be an oriented network such that each edge is incident with two distinct vertices. We suppose that with each edge there is associated a real number called the conductance of the edge.

We suppose the vertices of $N$ to be $p$ in number, and enumerate them as $P_1, P_2, \ldots, P_p$. Let $-c_{rs}$ ($r \neq s$) be the sum of the conductances of all the edges which are directed from $P_r$ to $P_s$, and let $c_{rr}$ be the sum of the conductances of the edges directed from $P_r$. Clearly

$$\sum_s c_{rs} = 0.$$ (7)

From (7) we can readily show that the cofactors of the elements of any particular row of the matrix $\{c_{rs}\}$ are all equal. We call their common value for the $r$th row the complexity of $N$ at $P_r$ and denote it by $C_r(N)$ or simply by $C_r$.

When $N$ has only one vertex $P_1$ we write $C_1(N) = 1$.

If it is also true that

$$\sum_r c_{rs} = 0,$$ (8)

we can likewise deduce that the cofactors of the elements of any particular column of $\{c_{rs}\}$ are all equal, whence it follows that $C_r$ has the same value $C(N) = C$ say for each $r$. We then call $C(N)$ the complexity of $N$.

Equation (8) is not true in general. We note, however, that it is true for the matrix obtained from $\{c_{rs}\}$ by multiplying the elements of each row by the corresponding $C_r$. The sum of the elements of any column of this matrix is equal to the determinant of $\{c_{rs}\}$ which, by (7), is 0. We call this matrix the standardized matrix of $N$. 
3.2. The second cofactor obtained by taking the cofactor of \(c_{rs}\) in the cofactor of \(c_{rt}\) (for \(r \neq s, t \neq u\)) is denoted by \((rs \cdot tu)\). We also write
\[
(rr \cdot tu) = (rs \cdot tt) = 0 \quad (\text{all } r, s, t, u).
\] (9)

From this definition we have
\[
(rs \cdot tu) = -(sr \cdot tu) = -(rs \cdot ut).
\] (10)

3.3. Consider the linear equations
\[
\sum_{u} c_{tu} V_u = \delta_{tr} H_r - \delta_{ts} H_s
\] (11)
in the unknowns \(V_u\) (\(\delta_{ii} = 1, \delta_{ij} = 0\) if \(i \neq j\)). We suppose \(r \neq s\). With respect to this set of equations we call \(P_r\) the positive and \(P_s\) the negative pole of \(N\). A necessary and sufficient condition for the consistency of equations (11) is that \(\{c_{rs}\}\) and the augmented matrix formed by adding to it a column whose \(t\)th element is \(\delta_{tu} H_r - \delta_{ts} H_s\) shall have the same rank\(^†\). For this it is necessary that the determinant of each square submatrix of order \(p\) of the augmented matrix shall vanish, i.e. that
\[
H_r C_r = H_s C_s.
\] (12)

If (12) is true, and if also \(C_s = 0\), the equations will be consistent, \(\{c_{rs}\}\) and the augmented matrix having the same rank \(p - 1\). If this is the case we can ignore the \(s\)th equation, which will be dependent upon the others. Multiplying each of equations (11) other than the \(s\)th by \(-1\) and adding to equation (7) multiplied by an arbitrarily fixed \(V_r\) we obtain a set of \(p - 1\) independent linear equations in the \(p - 1\) unknowns \(V_r - V_u\), where \(t\) is fixed and \(u \neq t\). The determinant \(D\) of this set of equations is the complementary minor of \(c_{st}\), whose value is \((-1)^{s+t} C_s \neq 0\). It follows that the \(p - 1\) equations have a unique solution. In this solution \(V_r - V_u\) is the cofactor of \(c_{ru}\) in the determinant \(D\), multiplied by \(-H_r\) and divided by \((-1)^{s+t} C_s\). That is
\[
V_r - V_u = \frac{-H_r}{C_s} (sr \cdot tu) = \frac{H_r}{C_s} (rs \cdot tu),
\] (13)

by (10).

From (13) we deduce the following polynomial identity (in the variables \(c_{ij}\)):
\[
(rs \cdot tu) + (rs \cdot uv) = (rs \cdot tv).
\] (14)

From the analogous result for the transpose of the standardized matrix of \(N\) we have also a polynomial identity
\[
C_s (qr \cdot tu) + C_q (rs \cdot tu) = C_r (gs \cdot tu).
\] (15)

It is of interest to compare these results with those of § 2.2 of (A). The fundamental distinction is that in (A) the matrix \(\{c_{rs}\}\) is symmetrical. Because of this we have for (A) the result \([rs \cdot tu] = [tu \cdot rs]\), but in the present theory it is not in general true that \((rs \cdot tu) = (tu \cdot rs)\). The theory reduces to that of (A) when we postulate that \(\{c_{rs}\}\) is symmetrical, and that two oriented edges of the same conductance \(c\) and with the same end-points but with opposite orientations are equivalent to a ‘wire’ of conductance \(c\). The interpretation of the complexity of an electrical network in terms of subtrees ((A), §(3.1)) also has a simple generalization in the present theory, as we now proceed to show.

\(^†\) A. C. Aitken, Determinants and matrices (Edinburgh, 1939), pp. 69–71.
3.4. As in (A) we define a subnetwork of $N$ as a network consisting of all the vertices and some subset of the edges of $N$. A subtree of $N$ is a subnetwork which is a tree, i.e. which is connected and which contains no simple closed curve. If the number of edges of the subtree $T$ of $N$ which have the vertex $P_r$ as positive (negative) end is 0 for a particular value $k$ of $r$ and 1 for every other value of $r$, then $T$ is said to converge to (diverge from) $P_r$.

We enumerate the subtrees of $N$ which converge to the vertex $P_k$, and denote by $\Pi_j$ the product of the conductances of the edges of the $j$th of them. We write

$$U_k(N) = \sum_j \Pi_j.$$  \hspace{1cm} (16)

3.5. Suppose that $N$ has at least one edge and at least three vertices. Let $P_j$ and $P_k$ be the positive and negative ends respectively of some edge $L$, of conductance $c$. Let $N'$ be derived from $N$ by suppressing $L$ and let $N''$ be derived from $N$ by suppressing all edges joining $P_j$ and $P_k$ and then identifying $P_j$ and $P_k$. From the definition of $C_k(N)$ we have

$$C_k(N') = (jk \cdot jk)$$  \hspace{1cm} (17)

and

$$C_k(N'') = C_k(N) - cC_k(N''),$$  \hspace{1cm} (18)

where $C_k(N'')$ is the complexity of $N''$ at the vertex obtained by identifying $P_j$ and $P_k$. It is clear that with an analogous interpretation of $U_k(N'')$ we have also

$$U_k(N'') = U_k(N) - cU_k(N'').$$  \hspace{1cm} (19)

3.6. Theorem. $C_k(N) = U_k(N)$.  \hspace{1cm} (20)

If $N$ has just one vertex $P_k$, $C_k(N) = 1$ (§ 3.1), but $U_k(N)$ is undefined. We define $U_k(N)$ to be 1 so that the theorem may be true in this case.

If $N$ has just two vertices $P_j$ and $P_k$, we have $C_k(N) = U_k(N) = -c_{jk}$. It is now only necessary to consider the case in which $N$ has at least three vertices.

If $P_k$ is not the negative end of any edge we have at once $U_k(N) = 0$. Moreover, with the possible exception of $c_{kk}$ the $k$th column of $\{c_{rs}\}$ consists entirely of 0's, so that $C_k(N)$, which can be defined as the cofactor of the $k$th element of another column, is 0. So the theorem is true in this case.

If $P_k$ is the negative end of an edge $L$, we define $N'$ and $N''$ as in § 3.5. By (18) and (19) the theorem will be true for $N'$ if it is true for $N'$', and $N''$ at $P_k$. As $N'$ and $N''$ each have fewer edges than $N$, the general result follows by induction over the number of edges of $N$.

3.7. We say that the network $N$ is simple if the conductance of each of its edges is 1 and if also each vertex has just as many edges directed to it as directed from it. Thus any simple $N$ satisfies (8) and so has the same complexity $C = C(N)$ at each vertex (by § 3.1).

3.8. Theorem. If $N$ is simple and connected, and has at least two vertices, and if $P_k$ is any one of its vertices, then $N$ has a subtree which converges to $P_k$.

Lemma. If $P_r$, $P_s$ are any two distinct vertices of $N$, then $P_r$ and $P_s$ can be joined in $N$ by a simple arc $\Lambda$ such that each vertex of $\Lambda$ other than $P_s$ is the positive end of just one edge in $\Lambda$. We call such an arc a directed arc from $P_r$ to $P_s$.

If the lemma is true for a particular pair $P_r$, $P_s$, we say that $P_s$ is accessible from $P_r$. 

W. T. TUTTLE
Let \( X \) be the set of all vertices of \( N \) accessible from \( P, \) together with \( P \) itself, and let \( Y \) be the set of all other vertices of \( N. \) If \( Y \) is not null, then since \( N \) is connected there must be at least one edge having one end in \( X \) and the other in \( Y. \) Clearly the end in \( X \) must be the negative end for each such edge (by the definitions of \( X \) and \( Y \)). It follows that the number of edges whose positive end is in \( Y \) exceeds the number whose negative end is in \( Y. \) Hence some vertex of \( Y \) is the positive end of more edges than have it as negative end, contrary to the definition of a simple network. Hence \( Y \) must be null and so the lemma is true.

Let the vertices of \( N \) other than \( P \) be enumerated as \( Q_1, Q_2, \ldots, Q_{p-1}. \) Let \( \Lambda_1, \Lambda_2, \ldots, \Lambda_{p-1} \) be directed arcs in \( N \) such that \( \Lambda_s \) is directed from \( Q_s \) to \( P. \) The existence of such arcs follows from the lemma.

We define \( G_1, G_2, \ldots, G_{p-1} \) successively as follows: \( G_1 = \Lambda_1. \) \( G_{s+1} (0 < s < p - 1) \) is the union of \( G_s \) and that part of \( \Lambda_{s+1} \) which extends from \( Q_{s+1} \) to the first vertex of \( \Lambda_{s+1}, \) reckoning from \( Q_{s+1}, \) which is in \( G_s \).

From this definition we find, by considering each \( G_s \) in turn, that \( G_s \) is a tree for each \( s. \) As \( G_{p-1} \) contains each vertex of \( N \) it is therefore a subtree of \( N. \) Also each vertex of \( G_s \) other than \( P \) is the positive end of just one edge of \( G_s, \) and \( P \) is not the positive end of any edge of \( G_s. \) Hence by §3·4, \( G_{p-1} \) converges to \( P. \)

**COROLLARY.** If \( N \) is simple and connected, then \( C(N) > 0. \)

This follows from §3·6.

3·9. From the above corollary, it follows that for simple networks we can replace (15) by

\[
(qr \cdot tu) + (rs \cdot tu) = (qs \cdot tu)
\]

in closer analogy with the equations of (A).

If \( N \) is simple we call the oriented network obtained from it by reversing the orientation of each edge the *reversal* of \( N, \) and denote it by \( N^*. \) Clearly \( N^* \) is simple.

If we distinguish quantities referring to \( N^* \) by an asterisk we have \( c_{rs}^* = c_{sr}, \) so that the matrix \( \{c_{rs}^*\} \) is the transpose of \( \{c_{rs}\}. \) From this it follows that

\[
C(N^*) = C(N) \tag{22}
\]

and

\[
(rs \cdot tu)^* = (tu \cdot rs). \tag{23}
\]

The reversal has no analogy in the theory of (A).

3·10. Consider the 2-complex \( N^\sigma(T) \) of §2·9. We define the conductance of each of its edges to be 1. Then the 1-section of \( N^\sigma(T) \) (i.e. the network defined by its edges and vertices) is simple, by §2·9. The quantity \( c_{rs}^\sigma \) for this network is clearly the quantity denoted in §2 by \( c_{rs}^\sigma. \)

Suppose that \( S^\sigma \) is \( p^\sigma \) and that \( \Sigma_\sigma \) is \( p^\sigma_\sigma. \) Then by applying §3·3 to equations (4) we deduce that

\[
V_r^\sigma - V_s^\sigma = \frac{X}{C^\sigma}(tu \cdot rs), \tag{24}
\]

where \( C^\sigma \) is the complexity of the 1-section of \( N^\sigma(T). \)

It is convenient to measure the size of \( \Delta \) in such units that \( X = C^\sigma. \) We then have

\[
V_r^\sigma - V_s^\sigma = (tu \cdot rs). \tag{25}
\]

We call the corresponding values of the \( x_r \) the *full sizes* of the elements of \( T \) with
4. CONSTRUCTION OF A TRIANGULATION FROM A BICUBICAL MAP

4.1. A bicubical map $M$ may be defined as follows. $M$ is a finite cellular 2-complex which is a dissection of $Z^2$, and which satisfies the following conditions:

(i) Each vertex is incident with just three 2-cells and therefore with just three edges, and

(ii) The vertices of $M$ fall into two mutually exclusive classes $E$ and $F$ such that each edge is incident with just one member of each class.

The bicubical map will be called admissible if it also satisfies the condition:

(iii) Each (closed) edge meets the boundaries of just four 2-cells.

Since the 2-cells are simple polygons it follows that the 1-section of $M$ is connected. The map $M(T)$ of §2 is an admissible bicubical map, by §2.7.

4.2. A 3-colouring of a bicubical map $M$ is a partitioning of its 2-cells among three mutually exclusive classes, called colour-classes, so that no two members of the same colour-class have a side in common.

**Theorem.** A bicubical map $M$ has just one 3-colouring.

Let the edges of $M$ be oriented so that the positive end of each is in $E$, and the 2-cells so that the 2-chain in which each coefficient is unity is a 2-cycle†. Then the 1-chain on $M$ in which the coefficient of each edge is the residue $1$ mod 3 is clearly a 1-cycle, $K^1$ say. Since $M$ is a 2-sphere this 1-cycle bounds a 2-chain $K^2$ over the additive group of residues modulo 3 on $M$. We classify the 2-cells of $M$ according to their coefficients in $K^2$. We thus obtain a 3-colouring of $M$, for if the two 2-cells incident with any edge have the same coefficient in $K^2$, that edge must have coefficient 0 in $K^1$.

It is easily seen that $M$ has at most one 3-colouring. For when the three 2-cells incident with any particular vertex are assigned to their colour-classes, the assignments at each of the vertices joined to the first vertex by a single edge are determined.

4.3. Consider an admissible bicubical map $M$. We enumerate the members of $E$ as $e_1, e_2, \ldots, e_n$ and the members of $F$ as $f_0, f_1, \ldots, f_{n-1}$. That $F$ has the same number of members as $E$ follows as in §2.3. We denote the three colour-classes of the bicubical map by $K(1), K(2), K(3)$ and enumerate the members of $K(\sigma)$ as $P_\sigma^i$ ($i = 1, 2, \ldots, m_\sigma$).

It will be seen that this notation agrees with that of §2 for the bicubical maps considered there. We define $L(\sigma)$ as in §2.6. We also define three 2-complexes $N^\sigma(M)$ just as we defined the $N(T)$ in §2.9. If $M$ is the bicubical map of a triangulation $T$ we can clearly suppose the notation adjusted so that $N^\sigma(M) = N^\sigma(T)$.

† For definitions of the terms of combinatorial topology used here, reference may be made to Seifert and Threlfall, *Lectures on Topology* (Leipzig and Berlin, 1934), to Alexandroff and Hopf, *Topologie* (Berlin, 1935), or to Lefschetz, *Algebraic Topology*, American Math. Soc. Colloquium publications, vol. 27. Here we use the results of Chapter V, §3 of the second of these works.
We suppose hereafter that the enumeration of the $P^r_\sigma$ is such that the member of $K(\sigma)$ incident with $f_0$ is $P^r_\sigma$ for each $\sigma$.

We define $\lambda^r_\sigma$ to be 1 if $P^r_\sigma$ is incident with $e_s$ and 0 otherwise.

4.4. Consider the equations

$$\sum \lambda^r_\sigma y_s = 0 \quad (\text{all } r \neq 1, \text{ all } \sigma). \quad (26)$$

$M$ has $2n$ vertices, $3n$ edges and $(n + 2)$ 2-cells (as in §2.12). Hence the equations (26) are $n - 1$ in number and involve just $n$ unknowns $y_s$. Since the equations are linear and homogeneous it follows that they have a solution in which the $y_s$ are real and not all zero. Henceforth by ‘the $y_s$’ we shall mean a particular solution of this kind. If $M = M(T)$ for some triangulation $T$ we get such a solution by putting $y_s = x_s$ ($f_0$ representing the exterior triplet) by §2.11.

4.5. We denote by $L^r_\sigma$ the edge of $L(\sigma)$ which is incident with $e_r$, and by $f^r_\sigma$ the member of $F$ which is incident with $L^r_\sigma$. Let

$$\Gamma^r = \sum_{\sigma, r} g^r_\sigma L^r_\sigma \quad (27)$$

be any 1-cycle on $M$ with rational integer coefficients $g^r_\sigma$ such that $g^r_\sigma = 0$ when $L^r_\sigma$ is incident with $f_0$. Orientation is defined as in §4.2.

Then $\Gamma^r$ bounds a 2-chain on $M$ in which the coefficients of the $P^r_\sigma$ are all equal, and therefore (by adding a 2-cycle) a 2-chain

$$H^r = \sum_{\sigma, r} h^r_\sigma P^r_\sigma \quad (28)$$

in which the $h^r_\sigma$ are rational integers such that $h^r_\sigma = 0$ whenever $r = 1$.

**Theorem.** For each 1-cycle $\Gamma^r$ of the form (27)

$$\sum_{\sigma, r} g^r_\sigma \omega^r y_s = 0, \quad (29)$$

where $\omega$ is an imaginary cube root of unity.

By the foregoing considerations it will suffice to prove this for the case in which $\Gamma^r$ bounds a 2-cell $P^r_\sigma$ not incident with $f_0$. For by (28) every $\Gamma^r$ of the form (27) is a linear combination of 1-cycles of this type.

Suppose then that $\Gamma^r$ bounds $P^r_\sigma$ and that the three colour-classes are $K(\rho)$, $K(\theta)$ and $K(\phi)$. Without loss of generality we can suppose that $g^r_\sigma$ is 0 when $L^r_\sigma$ is not incident with $P^r_\sigma$, and equal to $+1$ or $-1$ when $L^r_\sigma$ is incident with $P^r_\sigma$ according as $\sigma$ is $\theta$ or $\phi$. Then for $\Gamma^r$ we have, by (26),

$$\sum_{\sigma, r} g^r_\sigma \omega^r y_s = \sum_{\tau} \lambda^r_{\sigma, \tau} (\omega^\tau - \omega^\rho) y_s = (\omega^\tau - \omega^\rho) \sum_{\tau} \lambda^r_{\sigma, \tau} y_s = 0.$$

We have used the evident fact that edges of $L(\theta)$ must alternate with edges of $L(\phi)$ in the boundary of $P^r_\sigma$.

The theorem follows.

4.6. Let $A$ be any vertex of $M$ other than $f_0$. Then if $B$ is any other vertex of $M$ not $f_0$ we can, since $M$ is connected, find a 1-chain $Y$ whose combinatorial boundary is the 0-chain $B - A$. (We adopt the convention that the combinatorial boundary of $L^r_\sigma$ is $f^r_\sigma - e_r$.) We may suppose that the edges of $M$ incident with $f_0$ have zero coefficients.
in $Y$. We can arrange this if necessary by adding a suitable linear combination of the boundaries of the $P_k^r$.

Suppose that

$$Y = \sum_{\sigma, r} \nu_{\sigma}^{r} L_{\sigma}^{r}.$$  

Then we define the potential $\pi(B)$ of $B$ by

$$\pi(B) = \sum_{\sigma, r} \omega^r \nu_{\sigma}^{r} y_{\tau}.$$  

(30)

By equation (29) it follows that the same value of $\pi(B)$ is obtained for each possible $Y$. We note that $\pi(A) = 0$. Evidently differences of potential are independent of the choice of $A$.

Considering the edge $L_{\sigma}^{r}$ we find that

$$\pi(f_{\sigma}^{r}) - \pi(e_{\tau}) = \omega^r y_{\tau},$$  

(31)

provided that $f_{\sigma}^{r}$ is not $f_0$. If we try to calculate $\pi(f_{0})$ from the edge of $L(\sigma)$ which is incident with $f_0$ by (31), we shall obtain a result $\pi''(f_{0})$, but this will not necessarily have the same value for each $\sigma$. For convenience we also write, for each $r > 0$ and for each $\sigma$, $\pi(f_{\sigma}^{r}) = \pi''(f_{\sigma}^{r})$.

4.7. Let the complex numbers $\pi(e_{\tau})$, $\pi''(f_{\sigma}^{r})$ be represented by points in the Argand plane. The four points $\pi(e_{\tau})$, $\pi''(f_{\sigma}^{r})$, $\pi''(f_{\sigma}^{r+1})$ and $\pi''(f_{\sigma}^{r+2})$ coincide when $y_{\tau} = 0$, but otherwise the first is the centre of an equilateral triangle of which the other three are the vertices (by (31)). We denote this (closed) triangle by $E_{r}$ and call it an element.

The side of $E_{r}$ opposite $\pi''(f_{\sigma}^{r})$ is the set of points

$$\frac{1}{2} + \alpha) \pi''(f_{\sigma}^{r+1}) + (\frac{1}{2} - \alpha) \pi''(f_{\sigma}^{r+2}),$$  

(32)

where $\alpha$ takes all real values in the range $-\frac{1}{2} \leq \alpha \leq \frac{1}{2}$. The index $\sigma + 1$ or $\sigma + 2$, if greater than 3, is taken as equivalent to $\sigma - 2$ or $\sigma - 1$ respectively. Now (32) is the set

$$\pi(e_{\tau}) - \frac{1}{2} \omega^r y_{\tau} + (\omega^{\sigma+1} - \omega^{\sigma+2}) y_{\tau} \alpha,$$  

(33)

by (31). This is a segment of a straight line $Y + (\omega^{\sigma+1} - \omega^{\sigma+2}) \beta$, where $Y$ is a constant and $\beta$ is a real parameter. Let the value of $\beta$ at any point $\xi$ in this line be denoted by $\beta(\xi)$. Then evidently

$$\beta(\pi''(f_{\sigma}^{r+1})) - \beta(\pi''(f_{\sigma}^{r+2})) = y_{\tau}.$$  

(34)

Consider a particular 2-cell $P_k^r$ of $M$. For each $e_{\tau}$ incident with $P_k^r$ and such that $y_{\tau} \neq 0$, there is an element $E_{r}$. The sides of these $E_{r}$ opposite the vertices $\pi''(f_{\sigma}^{r})$ are parallel segments (by (33)). Moreover, they are in the same straight line and form a connected set. For if $f_{0}$ is not incident with $P_k^r$ each of these $E_{r}$ has a vertex in common with each of its neighbours in the cyclic sequence corresponding to the sequence of the $e_{\tau}$ with $y_{\tau} \neq 0$ in the boundary of $P_k^r$. If $f_{0}$ is incident with $P_k^r$ the same rule holds except that the two $E_{r}$ corresponding to the two $e_{\tau}$ on either side of $f_{0}$ need not have a vertex corresponding to $f_{0}$ in common. In either case the union of the sides of the $E_{r}$ opposite the vertices $\pi''(f_{\sigma}^{r})$ is connected, and so is a single straight segment; we denote it by $P_k^r$.

The segments $(0, \omega^r)$ and $(0, (\omega^{\sigma+1} - \omega^{\sigma+2}))$ are perpendicular. Hence by (33) if $y_{\tau} \neq 0$ the point $\pi(e_{\tau})$ and therefore the element $E_{r}$ lies on one side or the other of $P_k^r$ according as $y_{\tau}$ is positive or negative.

4.8. Let $\xi$ be any point in the Argand plane not contained in any of the segments $P_k^r$. Then we call the number of elements of which it is an interior point the degree of $\xi$. We denote this by $\delta(\xi)$. 
The segments $p^e$ are finite in number; the complement of their union must therefore have only a finite number of components. Let these be enumerated as $B_1, B_2, \ldots, B_n$, say. All the points of any particular one of these evidently have the same degree.

Consider a particular segment $p^e$ and let $\xi$ be a point in $p^e$ which is not the intersection of any two of the $p^e$. All but a finite number of the points of $p^e$ must be of this form. By (33) $p^e$ is a segment of a straight line $Y + (\omega^{e+1} - \omega^{e+2})\beta$, where $Y$ is a constant and $\beta$ is a real parameter.

Consider the cyclic sequence of the vertices in the boundary of $P^e$. Since edges of $L(\rho + 1)$ alternate with edges of $L(\rho + 2)$ in the boundary of $P^e$, we can suppose that each of the $e_i$ incident with $P^e$ is immediately succeeded in the cyclic sequence by $j^{e+1}$ and immediately preceded by $f^{e+2}$.

Now $\xi$ is in a side of $E_s$ if and only if either

(i) $\beta(\pi^{e+1}(j^{e+1})) > \beta(\xi) > \beta(\pi^{e+2}(j^{e+2}))$

or

(ii) $\beta(\pi^{e+1}(j^{e+1})) < \beta(\xi) < \beta(\pi^{e+2}(j^{e+2}))$.

$E_s$ is on one side or the other of $p^e$ according as $y_s$ is positive or negative (by §4.7), that is, according as (i) or (ii) holds (by (34)). But it is clear that the number of pairs $(f^{e+1}, j^{e+2})$ satisfying (i) is equal to the number satisfying (ii), save possibly in the case $i = 1$, when $P^e$ is incident with $f_0$. In that case if $\beta(\xi)$ lies between $\beta(\pi^{e+1}(j_0))$ and $\beta(\pi^{e+2}(j_0))$ the numbers of pairs satisfying (i) and (ii) will differ by 1; otherwise they will be equal.

Now for each $\sigma$ the segments $p^e$, $j^{e+1}$ have the point $\pi^{e+2}(f_0)$ in common. For if $e$ is the vertex of $E$ which is joined to $f_0$ by an edge of $L(\sigma + 2)$, $E_s$ has a side in each of $p^e$ and $j^{e+1}$ (definition of the $p^e$), and the vertex of $E_s$ opposite the third side is $\pi^{e+2}(f_0)$.

Hence either $p^e$, $j^{e+1}$ and $j^{e+1}$ contain the sides of an equilateral triangle $\Delta$ whose vertices are the points $\pi^e(f_0)$, or else they have a common point. In the latter case we say that $\Delta$ coincides with this point and has side zero.

There must be some point $\eta$ in a $B_s$ outside $\Delta$ such that $\delta(\eta) = 0$, since the outside of $\Delta$ is infinite in area. It follows by the above considerations that $\delta(\eta) = 0$ when $\eta$ is in a $B_s$ outside $\Delta$, and $\delta(\eta) = 1$ when $\eta$ is in a $B_s$ inside $\Delta$. Consequently the $E_s$ are the elements of a triangulation $T = T(M, f_0)$ of the equilateral triangle $\Delta$. (Since not all the $y_s$ are zero, $\Delta$ cannot in fact have side zero.)

4-9. The triangulation $T$ is not essentially altered if we multiply each $y_s$ by $-1$. For the effect of this on the potentials $n(B)$ is merely to multiply them by $-1$ (by (30)). The new $y_s$ still satisfy (26).

**Theorem.** We can arrange, by multiplying all the $y_s$ by $-1$, if necessary, that whenever $y_s = 0$, $y_s = x_s/\sqrt{3}$, $x_s$ being the size of the element $E_s$.

First, for any $E_s$ we find from (31) that

$$x_s = \pm |n(f^e) - n(j^{e+1})| = \pm y_s |\omega^e - \omega^{e+1}| = \pm \sqrt{3}y_s.$$

For any particular element $E_s$ we arrange, by changing the sign of all the $y_s$, if necessary, that $x_s$ and $y_s$ have the same sign.

But it is clear from (31) that all the positive elements have the same sign for $y_s$ and all the negative elements have the opposite sign for $y_s$. Hence $x_s$ and $y_s$ now have the same sign for each $E_s$.

We assume henceforth that $y_s$ is made to satisfy this condition.
4.10. We define a matrix \( c_{rs}^\sigma \) for \( N^\sigma(M) \) just as we defined it for \( N^\sigma(T) \) in \$2.10. Equations (1) and (2) then hold for \( N^\sigma(M) \).

We define \( V_r^\sigma \) to be \( 2/\sqrt{3} \) times the distance of \( p_r^\sigma \) from \( p_0^\sigma \) measured positively towards \( \pi^\sigma(f_0) \).

It is easily verified, by an argument similar to that of \$2.11, that equations (4) and (5) are also true for \( N^\sigma(M) \). Applying the theory of \$3 (as in \$3.10) we then find that the new quantities \( V_r^\sigma \) satisfy (24), \( C^\sigma \) and \( (tu \cdot rs) \) being defined in terms of \( N^\sigma(M) \). It follows that when \( X \) is given, the differences \( V_r^\sigma - V_s^\sigma \) and therefore the quantities \( x_r \) and so also the \( y_r \) are fixed uniquely. Thus the solution of (26) for the \( y_r \) is unique apart from multiplication by an arbitrary constant.

When we measure the side of \( \Delta \) in such units that \( X = C^\sigma \), (24) reduces to (25), and so the sizes of the elements become integers. We have still to prove that \( C^\sigma \) is the same for each \( \sigma \).

\[ \text{Fig. 1.} \]

4.11. From a given bicubical map \( M \) we can in general derive several distinct triangulations by taking different members of \( F \) as \( f_0 \). Further, we can interchange the members of \( E \) and \( F \) and then derive another set of triangulations. This operation evidently replaces \( N^\sigma(M) \) by its reversal.

It is therefore possible to determine all the triangulations of equilateral triangles of a reasonably low order \( n \) by first listing the bicubical maps of \( 2n \) vertices and then deriving from each the corresponding triangulations. Alternatively, we may prefer to list the maps \( N^\sigma(M) \), characterized by the property that at each vertex edges directed to that vertex alternate with edges directed from it. (It is easily verified that the structure of \( N^\sigma(M) \) completely determines that of \( M \).) This is in closer analogy with the methods of (A). Not all the triangulations so obtained will be of the \( n \)th order necessarily, since in particular cases some of the \( y_r \) may vanish, but any triangulation \( T \) of the \( n \)th order will clearly be obtained from the corresponding map \( M(T) \). I find that the two simplest perfect triangulations of equilateral triangles are those which can be obtained from the parallelogram of Fig. 1 by erecting equilateral triangles on two of its sides meeting at an acute angle†.

† In Figs. 1–3, any pair of elements having a common side is represented as a rhombus. The numbers represent the lengths of the sides of the containing polygons, or of the dissected figure.
5. Triangulations of parallelograms

5.1. The theory of triangulations of equilateral triangles is easily modified to cover that of triangulations of parallelograms (with angles of $\frac{1}{3}\pi$ and $\frac{2}{3}\pi$). I find that the simplest perfect parallelogram of order $n > 2$ is that of Fig. 1.

Given any triangulation $T$ of order $n$ of a parallelogram $\Pi$ we can erect two new elements $E_{n+1}, E_{n+2}$ on two of its sides adjacent to an acute angle and so obtain a triangulation $T'$ of order $n+2$ of an equilateral triangle $\Delta$.

For $\Delta$ we will define $S_1$ to be the side meeting both $E_{n+1}$ and $E_{n+2}$. Then in $M(T')$ the 2-cell $Q$ corresponding to $S_1$ is a quadrilateral, for it is incident with $e_{n+1}$ and $e_{n+2}$ but with no other member of $E$. One of the members of $F$ incident with $Q$ is the representative vertex $f_0$ of the exterior triplet.

Conversely, with the notation of §4, suppose that $P_1$ is a 2-cell of $M$ which is a quadrilateral, and which is incident with $f_0$. Let the three members of $E$ joined to $f_0$ by 1-simplexes be $e_r, e_{r+1}$ and $e_{r+2}$, the two latter being incident with $P_1$. Let the member of $K(2)$ on the opposite side of the quadrilateral $P_1$ to $P_2$ be $P_3$. Then the triangulation $T(M, f_0)$, the side $S_1$ of $\Delta$ corresponding to $P_1$, contains sides of just two elements $E_{r+1}$ and $E_{r+2}$ of $T(M, f_0)$. Suppressing these two elements we obtain a triangulation of a parallelogram. The lengths of the sides of this parallelogram can be obtained in terms of $N^2(M)$ by using (24) with the interpretation of §4·10. If we adopt the convention that the side of $\Delta$ is $C^2$ (the complexity of $N^2(M)$), they are $(1j\cdot 1j)$ and $C^2 - (1j\cdot 1j)$.

5.2. We can obtain another triangulation of a parallelogram ($\Pi^*$ say) by interchanging the members of the classes $E$ and $F$ and then taking the vertex denoted above by $e_{r+1}$ to represent the exterior triplet of a corresponding triangulation of a triangle. For the quadrilateral $P_1$ represents a side of the triangulated triangle. In this operation on $M, N^2(M)$ is replaced by its reversal. As before, if the side of the triangle is $(C^2)^*$, the complexity of $(N^2(M))^*$, then the sides of $\Pi^*$ are $(1j\cdot 1j)^*$ and $(C^2)^* - (1j\cdot 1j)^*$.

Thus the sides of $\Pi^*$ have the same lengths as those of $\Pi$ (by (22) and (23)). In general, therefore, given any triangulation of a parallelogram $\Pi$, we can find another triangulation of $\Pi$ of the same or smaller order. (Conceivably $(rs \cdot tw)^*$ but not $(rs \cdot tw)$ may vanish). Figs. 2 and 3 show two perfect triangulations related in this way.

5.3. Evidently to any 2-cell of $M$ which is a quadrilateral not incident with $f_0$ there corresponds a pair of elements with a common side in the triangulation $T(M, f_0)$.

A case of particular interest arises when all the members of $K(1)$ are quadrilaterals. In any corresponding triangulated parallelogram the elements then fall into disjoint pairs, each pair constituting a rhombus, and the shorter diagonals of the rhombuses are all parallel. Such a dissection of a parallelogram into rhombuses is clearly equivalent to a dissection of a rectangle of the same side-lengths into squares. Conversely, by 'shearing' any squared rectangle we can obtain a dissection of a parallelogram into rhombuses, and we can relate this to an admissible bicubical map $M$ in which all the members of $K(1)$ are quadrilaterals.
It is easily verified that in this case $N^2(M)$ and $N^3(M)$ correspond to dual c-nets† associated with the squared rectangle. A 'wire' in such a c-net is represented by two oppositely directed edges with the same end-points and bounding a 2-sided 2-cell in the corresponding map $N^2(M)$ or $N^3(M)$. This 2-cell is a quadrilateral of $K(1)$ in $M$.

The quantities ($rs$. $tu$) for $N^2(M)$ and $N^3(M)$ become identical with the transiances of the corresponding c-nets.

This is why we describe the relationship between the three maps $N^r(M)$ as a generalization of topological duality on the sphere. Further justification is given by the theorem of §6.

† (A) §(3·3).
Dr C. A. B. Smith points out that the results of (A) enable us to prove the following
Theorem. A regular hexagon can be dissected into rhombuses, all of different sizes.

We first dissect the hexagon into three equal rhombuses by joining the centre to alternate vertices. We obtain dissections of two of these by 'shearing' perfect squares.
If the two perfect squares are 'totally different' ((A), § (8·1)) the hexagon will now be
dissected into unequal rhombuses (see (A), § (9·11)).

Corollary. There exists a perfect triangulation of a regular hexagon.

We dissect the hexagon into unequal rhombuses and then dissect each rhombus into
two equilateral triangles. One of these triangles will be a positive, the other a negative
of the triangulation.

6. THE COMPLEXITIES $C^a$

6·1. In § 4 we showed that in the triangulation $T(M, f_0)$ obtained from a bicubical
map $M$ the sizes of the elements all become integers when the side of the complete
triangle is made equal to $C^a$. This suggests that

$$C^1 = C^2 = C^3.$$  \hspace{1cm} (35)

The object of the present section is to prove (35) for every bicubical map $M$. Equation
(35) is the analogue of Theorem (3·25) of (A). This applies to dual undirected networks,
whereas (35) applies to trial directed ones.

6·2. In any bicubical map $M$ members of the three classes $K(1), K(2)$ and $K(3)$ occur
in the same cyclic order at each member of $E$, and in the opposite cyclic order at each
member of $F$ (with respect to a fixed positive sense of rotation in $Z^3$). To prove this we
have only to observe that the orders at the two ends of any given edge must be opposite.

6·3. Select any particular member $e_k$ of $E$.

Suppose that there are given sets $D^1, D^2, D^3$ of edges of $N^1(M), N^2(M)$ and $N^3(M)$
respectively, together forming a set $D$ such that (i) if $P$ is any 2-cell of $M$ not incident
with $e_k$, then just one member of $D$ has $P$ as its positive end in the appropriate $N^a(M)$,
(ii) if $P$ is any 2-cell of $M$ incident with $e_k$, then $P$ is not the positive end of any member
of $D$ in an $N^a(M)$, (iii) each member of $E$ other than $e_k$ is the positive end in $M$ of just
one member of $D$, and (iv) $e_k$ is not incident in $M$ with any member of $D$.

We denote by $R^a$ the subnetwork (see § 3·4) of the 1-section of $N^a(M)$ whose edges
are the members of $D^a$. (We call it also a subnetwork of $N^a(M)$.)

Theorem. For each $D$ satisfying the above conditions, and for each $a$, $R^a$ is a subtree
of $N^a(M)$ converging to that vertex of $N^a(M)$ which, as a 2-cell of $M$, is incident with $e_k$.

We suppose this vertex denoted by $P^a_\Gamma$ (contrary to our former convention).

Assume that for some $a$, $R^a$ contains a simple closed curve $\Gamma$.

Then each vertex of $N^a(M)$ in $\Gamma$ is the positive end of just one edge of $\Gamma$. Otherwise
some vertex will be the positive end of two edges of $\Gamma$, contrary to (i) and (ii). Hence
by (ii) $e_k$ is not incident in $M$ with any vertex of $\Gamma$.

Clearly by making joins inside some 2-cells of $M$ (vertices of $\Gamma$) we can obtain a
simple closed curve $\Gamma'$ in $M$ which contains every edge of $M$ which is an edge of $\Gamma$ and
otherwise lies entirely in the interiors of the 2-cells of $K(a)$. By the preceding paragraph,
\( \Gamma' \) does not contain \( e_k \), and members of \( E \) and \( F \) occur alternately in \( \Gamma' \) (i.e. the edges of \( M \) in \( \Gamma' \) are all directed the same way round \( \Gamma' \)).

From this, with the help of (6.2) we deduce that the 2-cells of \( M \) contained in a particular residual domain \( X \) of \( \Gamma' \) and meeting \( \Gamma' \) all belong to the same colour-class \( K(\rho) \) say, which is not \( K(\sigma) \). We will take \( X \) to be that residual domain of \( \Gamma' \) which does not contain \( e_k \).

Now any edge of \( D^\sigma \) having its positive end (as an edge of \( N^\rho(M) \)) contained in \( X \) also has its negative end (as an edge of \( N^\rho(M) \)) contained in \( X \) and is itself contained in \( X \). Otherwise it would have to be incident with a vertex of \( M \) contained in \( \Gamma' \). This vertex would have to belong to \( F \) (by (i) and (ii)), which would imply that some 2-cell \( \dagger \) contained in \( X \) and not of colour \( \sigma \) or \( \rho \) would meet \( \Gamma' \).

Each edge of \( M \) in \( \Gamma' \) is incident with a 2-cell contained in \( X \). Hence \( X \) contains at least one 2-cell of \( K(\rho) \).

Let \( H \) be the part of \( N^\rho(M) \) consisting of those edges and vertices which, as edges and 2-cells of \( M \), are contained in \( X \). Let \( H_0 \) be obtained from \( H \) by suppressing all the edges except those whose positive ends are in \( X \) and which belong to \( D^\rho \). By the preceding considerations \( H_0 \) is a network which with any vertex of \( N^\rho(M) \) contains also the edge, if any, of \( D^\sigma \) which has that vertex as positive end. Since \( e_k \) is not in \( X \), it follows from (i) that \( H_0 \) contains precisely as many edges as vertices. So, using the formula
\[
p_1(L) - p_0(L) = \alpha_1(L) - \alpha_0(L),
\]
(36)
\( L \) being any network, \( p_i(L) \) the Betti number of dimension \( i \) of \( L \), and \( \alpha_1(L) \) and \( \alpha_0(L) \) the numbers of edges and vertices of \( L \) respectively, we see that \( p_1(H_0) - p_0(H_0) > 0 \), so that \( H_0 \) must contain a simple closed curve.

We thus deduce that, given any simple closed curve \( \Gamma' \) in \( M \) containing one or more edges of \( D^\sigma \), for some particular \( \sigma \), and otherwise lying entirely in the interiors of the 2-cells of \( K(\sigma) \), we can find another such curve, corresponding to a different value of \( \sigma \), which is separated from \( e_k \) by \( \Gamma' \). But this implies the existence of an infinity of such curves, of which no two intersect. This contradicts our requirement that the edges of \( M \) are finite in number.

It follows that \( R^\sigma \) has no simple closed curve. It has just one more vertex than edge (by (iii) and (iv)). Hence by (36), \( p_0(R^\sigma) = 1 \), and so \( R^\sigma \) is connected. Hence \( R^\sigma \) is a subtree of \( N^\sigma(M) \). Another application of (i) and (ii) shows that \( R^\sigma \) converges to \( P^\sigma_1 \).

6.4. Theorem. Let \( R^\sigma \) be any subtree of \( N^\sigma(M) \) converging to \( P^\sigma_1 \). Then there is just one set \( D \) satisfying (i)–(iv) such that the edges belonging to \( D^\sigma \) are the edges of \( R^\sigma \).

We define a sequence \( Y(0), Y(1), Y(2), \ldots \) of sets of edges of \( M \) in the following way.

We first agree that any 2-cell \( P \) of \( M \) not incident with any member of \( E \) which is the positive end of a member of \( Y(i) \) of the same colour as \( P \) shall be called \( i \)-unused. Further we agree that any member of \( E \) which is incident with no member of \( Y(i) \) shall be called \( i \)-unused.

Given \( Y(i) \) we choose, if possible, an \( i \)-unused 2-cell \( P \), not incident with \( e_k \), which is

\dagger The 2-cell is that one incident with the given edge of \( D^\sigma \) and not of colour \( \sigma \).
\dagger Seifert and Threlfall, p. 87 (Example 3).
incident with just one $i$-unused member $e_i$ of $E$. We adjoin to $Y(i)$ the edge incident with $e_i$ and having the same colour as $P$. We take the resulting set as $Y(i+1)$. We define $Y(0)$ to be the set of edges of $R^\sigma$ and continue the above process until it terminates, with $Y(m)$ say. It follows from (i) that any set $D$ which contains $Y(0)$ contains also $Y(m)$.

All the 0-unused, and therefore all the $m$-unused 2-cells of $M$ not incident with $e_k$ have colours other than $\sigma$, since $R^\sigma$ is a subtree of $N^\sigma(M)$.

We can show that each $m$-unused member of $E$ other than $e_k$ is incident with just two $m$-unused 2-cells. For let $e_j$ be any $m$-unused member of $E$ other than $e_k$ ($e_k$ is incident with three $m$-unused 2-cells). $e_j$ is incident with just two 2-cells, $P_1$ and $P_2$ say, of colours other than $\sigma$. If either of these is incident with $e_k$ it is $m$-unused by the construction. If one of them, $P_1$ say, is not incident with $e_k$ and not $m$-unused, then for some $i < m$ it is $i$-unused but not $(i + 1)$-unused. Hence $Y(i+1)$ was formed from $Y(i)$ by adding an edge incident with an $i$-unused vertex $e_\ell$ incident with $P_1$ and not identical with $e_j$. Since $e_j$ is also $i$-unused and incident with $P_1$ this was contrary to the conditions of the construction.

Construct a network in which the vertices are the $m$-unused members of $E$, together with a set consisting of one interior point from each $m$-unused 2-cell other than $P^\sigma_1$, and in which the edges are simple arcs contained in the $m$-unused 2-cells. In each $m$-unused 2-cell there is just one such arc for each incident $m$-unused member of $E$ joining this member of $E$ to the chosen interior point. We can suppose that no two of these arcs in any one 2-cell intersect. Denote this network by $G$. Let $G'$ be obtained from $G$ by suppressing all the isolated vertices. Let the number of $m$-unused vertices of $M$ be $p$, and the number of other vertices of $G'$ be $q$. Then the number of edges of $G'$ is $2p$. Also it is not less than $2(q-2)+2=2q-1$ with equality possible only when no 2-cell incident with $e_k$ and not of colour $\sigma$ is incident with any other $m$-unused member of $E$ and when also no $m$-unused 2-cell is incident with more than two $m$-unused members of $E$. (The $P^\sigma_1$ are the only 2-cells which can be at once $m$-unused and incident with just one $m$-unused member of $E$; otherwise we could construct a $Y(m+1)$.) Hence

$$p \geq q-1$$

with equality only under the above conditions.

Now $G'$ contains no simple closed curve. For if it contains a simple closed curve $\Gamma$, each component of the complement of $\Gamma$ in $Z^2$ contains a member of $F$, and therefore a 2-cell of colour $\sigma$. The union of the (open) 2-cells of $K(\sigma)$ and the closures in $M$ of the edges which are edges of $R^\sigma$ is a connected set ($R^\sigma$ is connected). But it does not meet $\Gamma$. This is absurd since we have shown that it meets each component of the complement of $\Gamma$. Hence if the number of components of $G'$ is $p_0(G')$, we can by applying (36) to $G'$ deduce that

$$2p = (p+q)-p_0(G').$$

Hence $p_0(G') = q - p \leq 1$ (by (37)). As $p_0(G')$ must exceed 0, since $G'$ certainly contains $e_k$, it follows that $p = q - 1$. We have seen that this can be true only when $e_k$, together with its two incident edges of $G'$ constitutes a complete component of $G'$. Since $p_0(G') = 1$ this component is the whole of $G'$. Hence we have the result

$M$ contains no $m$-unused member of $E$, other than $e_k$. 

Dissection of equilateral triangles into equilateral triangles 481
Now if $M$ has $n$ members of $E$, the number of its 2-cells is $n + 2$, by (4·4) and (2·12). It follows by the method of construction of $Y(m)$ that the number of $m$-unused 2-cells is $(n + 2) - (n - 1) = 3$. These three are of course those incident with $e_h$.

It follows that $Y(m)$ satisfies the conditions (i)–(iv). But we have shown that any set $D$ satisfying (i)–(iv) and containing $Y(0)$ contains also $Y(m)$. The theorem follows.

6·5. By the results of §§ 6·3 and 6·4 we can arrange the subtrees of the maps $N^\rho(M)$ converging to the vertices $P^*_1$ in disjoint sets of three so that just one member of each set is a subtree of $N^\rho(M)$ for each $\rho$. The three members of any one of these sets define a set $D$ satisfying (i)–(iv). (35) follows immediately by § 3·6.

We have now obtained the following curious topological theorems.

(i) The number of subtrees of an $N^\rho(M)$ converging to a fixed vertex is independent of the particular vertex chosen. (§ 3·7.)

(ii) The number of subtrees of an $N^\rho(M)$ converging to a given vertex is equal to the number diverging from that vertex (by (22)).

(iii) The numbers of subtrees of $N^1(M)$, $N^2(M)$ and $N^3(M)$ converging to particular vertices $P^1$, $P^2$, $P^3$ respectively, are equal.